

# Modular Termination of r-Consistent and Left-Linear, Term Rewriting Systems

Manfred Schmidt-Schauß<sup>1</sup>, Massimo Marchiori<sup>2</sup>,  
Sven Eric Panitz<sup>1</sup>

<sup>1</sup> Fachbereich Informatik, J.W.Goethe-Universität Frankfurt,  
PO-Box 111932, Robert-Mayer-Str 11-15, 60054 Frankfurt, Germany  
email: schauss@informatik.uni-frankfurt.de

<sup>2</sup> Department of Pure and Applied Mathematics, University of Padova,  
Via Belzoni 7, 35131 Padova, Italy  
email: max@hilbert.math.unipd.it

## Abstract.

A modular property of term rewriting systems is one that holds for the direct sum of two disjoint term rewriting systems, iff it holds for every involved term rewriting system. A term rewriting system is r-consistent, iff there is no term that can be rewritten to two different variables. We show that the subclass of left-linear and r-consistent term rewriting systems has the modular termination property. This subclass may also contain nonconfluent term rewriting systems. Since confluence implies r-consistency, this constitutes a generalisation of the theorem of Toyama, Klop, and Barendregt on the modularity of termination for confluent and left-linear term rewriting systems.

## Acknowledgements.

We thank Bernhard Gramlich, Jan Willem Klop and Enno Ohlebusch for valuable comments on a draft of this paper.

## 1. Introduction.

It is well-known that the direct sum of two disjoint term rewriting systems may be nonterminating even if every single system is terminating. The counterexamples which are presented in [To87a] demonstrate that several naïve conjectures were false. For example, it is shown that left-linearity alone is not sufficient for termination, and furthermore, that convergence, i.e. termination and confluence, of a term rewriting system is not sufficient for termination of their direct sum.

A simple example is :

$$R_1: \quad f(1,2,x) \rightarrow f(x,x,x)$$

$$R_2: \quad G(x,y) \rightarrow x, \quad G(x,y) \rightarrow y$$

Both systems are terminating, however,  $f(G(1,2), G(1,2), G(1,2))$  starts a cyclic reduction.

This example was a natural hurdle for finding sufficient criteria for termination of the direct sum. In [TKB89] it is proved that for left-linear and confluent term rewriting systems the termination is inherited to the direct sum. In the example above  $R_1$  and  $R_2$  are both left-linear, but  $R_2$  was not confluent.

The term rewriting system  $R_2$  is not r-consistent, since we can rewrite  $G(x,y)$  to  $x$  as well as  $y$ .

A further counterexample in [Dr89] (which is a bit simpler than that in [To87a]) is

$$\begin{array}{l}
 R_3: \quad f(0,1,x) \quad \rightarrow f(x,x,x) \\
 \quad \quad f(x,y,z) \quad \rightarrow 2 \\
 \quad \quad 0 \quad \quad \rightarrow 2 \\
 \quad \quad 1 \quad \quad \rightarrow 2 \\
 R_4: \quad G(x,y,y) \quad \rightarrow x \\
 \quad \quad G(y,y,x) \quad \rightarrow x
 \end{array}$$

Both systems  $R_3$  and  $R_4$  are convergent, however, their direct sum is not terminating. We have  $G(0,1,1) \rightarrow 0$  and  $G(0,1,1) \xrightarrow{*} G(2,2,1) \rightarrow 1$ . There is a cyclic reduction  $f(G(0,1,1), G(0,1,1), G(0,1,1)) \xrightarrow{*} f(0, 1, G(0,1,1)) \xrightarrow{*} f(G(0,1,1), G(0,1,1), G(0,1,1))$ .

This example combines a left-linear and convergent TRS with a non left-linear and convergent one. Thus convergence alone is also not sufficient for the modularity of termination.

There are further results on sufficient criteria for termination ([Ru87, Mi87], see also [Kl92]), such as left-dominance of both term rewriting system, where left-dominance means that the number of variable occurrences of every variable does not increase in any reduction step. Furthermore, if every system is collapse-free, i.e., there are no rules of the form  $t \rightarrow x$ , then termination is modular. In the example above,  $R_3$  is not left dominant.

A motivation for studying termination properties of direct sums of term rewriting systems is that they can be considered as a model for functional programming using recursion equations. The direct sum of term rewriting systems corresponds to the usage of independently defined modules. The issue is, whether the evaluation of an expression terminates, if all functions from the modules terminate. In the cases where all the recursion equations are restricted to being nonoverlapping and left-linear, i.e., they are orthogonal, they are also confluent (see [Hu80]), hence the theorem of [TKB89] can be used.

In this paper, we show that the theorem from [TKB89] that states modularity of termination for confluent, left-linear term rewriting systems can be generalised to  $r$ -consistent term rewriting systems. Furthermore, the proof of this result is simplified.

## 2. Preliminaries.

We use a lot of standard notions and notations, (cf. [DJ90], [Kl92]).

Let  $\text{Term}(F,V)$  be the set of terms defined over a set of function symbols, where every function comes with an arity, and a set  $V$  of variables. The set of variables occurring in a term  $t$  is denoted by  $\text{Var}(t)$ . A term is linear, if every variable occurs at most once in it. The set  $O(t)$  is the set of occurrences of a term  $t$ . The subterm of  $t$  at position  $p$  is denoted by  $t/p$ . The term  $t[p \rightarrow s]$  is obtained from  $t$  by replacing the subterm at occurrence  $p$  by  $s$ . Two occurrences  $p, q$  are disjoint (independent) (notation  $plq$ ), if neither  $p$  is below  $q$  nor  $q$  is below  $p$ . If  $p$  is a prefix of  $q$ , we denote this by  $p \geq q$  ( $p > q$  if  $p \neq q$  in addition). In this case we also say  $q$  is below  $p$  ( $p$  is above  $q$ ). We define  $\text{root}(t)$  to be the function symbol at the top, if  $t$  is not a variable and  $\text{root}(t) = t$ , if  $t$  is a variable.

A substitution  $\sigma$  is a mapping  $\sigma: \text{Term}(F,V) \rightarrow \text{Term}(F,V)$  such that  $\text{Dom}(\sigma) := \{x \in V \mid \sigma x \neq x\}$  is finite and such that  $\sigma(f(t_1, \dots, t_n)) = f(\sigma t_1, \dots, \sigma t_n)$  for all terms  $f(t_1, \dots, t_n)$ . A substitution can be represented by its values on  $\text{Dom}(\sigma)$ :  $\sigma = \{x_1 \rightarrow \sigma x_1, \dots, x_n \rightarrow \sigma x_n\}$ .

A rewrite rule is a pair of terms, denoted  $l \rightarrow r$ , such that  $\text{Var}(r) \subseteq \text{Var}(l)$  and  $l$  is not a variable. A term rewriting system is a set of rewrite rules. The reduction relation  $\rightarrow_R$  for a term rewriting system  $R$  is defined as follows:  $s \rightarrow_R t$ , iff there is a position  $p \in O(s)$ , a rule  $l \rightarrow r$  in  $R$ , and a substitution  $\sigma$ , such that  $\sigma l = s/p$ , and  $t = s[p \rightarrow \sigma r]$ , the transitive, reflexive closure is denoted by  $\xrightarrow{*}_R$ , the transitive closure by  $\xrightarrow{+}_R$ . The symmetric, reflexive and transitive closure is denoted by  $=_R$ , and also called the equational theory generated by  $R$ . We say  $R$  is consistent, if  $x =_R y$  doesn't hold for two different variables. Furthermore, we say  $R$  is consistent w.r.t. reduction ( $r$ -consistent), if there is no term  $t$ , such that  $t \xrightarrow{*}_R x$  and  $t \xrightarrow{*}_R y$  for different variables  $x$  and  $y$ .

A term rewriting system  $R$  (the corresponding reduction relation  $\rightarrow_R$ ) is called Noetherian (or terminating), if there are no infinite reduction sequences, it is called confluent, whenever  $v \xrightarrow{*}_R t \xrightarrow{*}_R w$  implies that there exists a term  $s$ , such that  $v \xrightarrow{*}_R s \xleftarrow{*}_R w$ .

Note that for term rewriting systems we have that confluence implies consistency which in turn implies  $r$ -consistency.

If a term rewriting system is Noetherian and confluent, then it is also called convergent. In a convergent term rewriting system, every term  $t$  can be reduced to a unique normal form, denoted by  $t \downarrow_R$ .

A rule  $l \rightarrow r$  is called left-linear if  $l$  is linear, a term rewriting system is called left-linear if all rules are left-linear. A rule is collapsing, if it is of the form  $l \rightarrow x$  for some variable  $x$ . A TRS is collapsing, if it contains at least one collapsing rule. A rule  $l \rightarrow r$  is called non-erasing, iff  $\text{Var}(r) = \text{Var}(l)$ , a term rewriting system is called non-erasing, iff all rules in it are non-erasing. Note that in other references this property is also called regular or variable-preserving.

Let  $R_i$ ,  $i = 1, 2$  be term rewriting systems for  $\text{Term}(F_i, V)$ , where the corresponding sets of function symbols  $F_1$  and  $F_2$  are disjoint. In this case we call  $R_1$  and  $R_2$  disjoint. The direct sum  $R_1 \oplus R_2$  of two disjoint term rewriting systems is defined as the the union of the rules, and the reduction relation is defined on the set  $\text{Term}(F_1 \cup F_2, V)$ . A property  $P$  is called modular, iff  $P(R_1)$  and  $P(R_2)$  is equivalent to  $P(R_1 \oplus R_2)$ . If a modular property  $P$  holds for several disjoint term rewriting systems, it can be lifted to their direct sum.

Let  $t$  be a term in  $\text{Term}(F_1 \cup F_2, V)$ . We say the theory of a nonvariable term  $t$  is  $i$ , ( $\text{Th}(t) = i$ , or  $t$  is an  $i$ -term) if  $\text{root}(t) \in F_i$ . We say a subterm  $s = t/p$  is alien, if  $p \neq \varepsilon$  ( $\varepsilon$  denotes the empty word) and if for the direct superterm  $s_0 = t/p_0$  with  $p = p_0.j$  we have  $\text{Th}(s) \neq \text{Th}(s_0)$ . We say a subterm  $s$  of  $t$  is a maximal alien subterm, if there is no alien proper super-term of  $s$  in  $t$ . Notation is simplified by using contexts, denoted by  $t[[t_1, \dots, t_n]]$ . This can be formalized as a term  $t$  with  $n$  holes at occurrences  $p_1, \dots, p_n$ , and  $t[[t_1, \dots, t_n]]$  is the term  $t[p_1 \rightarrow t_1, \dots, p_n \rightarrow t_n]$ . Given some term  $s_1$ , and some alien subterm  $s_0$  denoted as a context  $s[[t_1, \dots, t_n]]$ , where  $t_i$ ,  $i=1, \dots, n$  are all the maximal alien subterms of  $s_0$ , we will call  $s[[\dots, \dots, \dots]]$  a layer within  $s_1$ . The top layer means the topmost layer of some term. The theory-length of a position  $p$  in  $t$  is the number of theory changes on a path from the root to this position including the term  $t/p$ . The rank of a term  $t$  is defined as the maximal theory-length of a position of a variable or a constant in  $t$ .

We need some non standard notions that are suitable for reduction with left-linear term rewriting systems. Therefore we give also a rigorous definition of the notion of term.

## 2.1 Definition. Labelled terms.

A subset  $\emptyset \neq T \subseteq \mathbb{N}^*$  is an ordered tree, iff i)  $T$  is prefix-closed, i.e. the prefix of every word in  $T$  is also in  $T$ , and ii)  $p.i \in T \wedge i > 1 \Rightarrow p.(i-1) \in T$ .

Let  $F$  be a signature,  $\text{arity}: F \rightarrow \mathbb{N}$  be a function, and  $V$  be an infinite set of variables. A term  $t$  is a finite ordered tree  $T$  together with a function  $t: T \rightarrow F \cup V$ , such that  $t(p) = f \in F$  implies that  $p$  has  $\text{arity}(f)$  sons;  $t(p) \in V$  implies that  $p$  has no sons.

Let  $L$  be a set of labels.

A **labelled term** consists of an ordered tree  $T$ , a function  $t: T \rightarrow F \cup V$  and a partial function  $m: T \rightarrow L$ . I.e., the nodes of a term are not only marked by function symbols or variables, but additionally nodes may be labelled.  $\blacklozenge$

The labels at the nodes of a term are not intended to contribute to the structure of the term, but only to record the changes that rewriting does. Hence rewriting, equality tests and the like only look for the term part of a labelled term. However, there is one difference, a rewriting step has to label the resulting term in a sensible way. In particular, we want to give subterms a form of identity that survives rewriting, which is only possible for left-linear term rewriting systems,

**2.2 Definition.** Let  $R_1$  and  $R_2$  be disjoint, left-linear term rewriting systems. We consider rewriting of labelled terms in  $R_1 \oplus R_2$ .

Let  $t$  be a labelled term  $t$  with labelling  $m$ . For the purposes of this paper it is sufficient to assume that only the root of alien subterms, the root of  $t$  or the variables are labelled.

The operation of a left-linear rule  $l \rightarrow r$  on  $t$  at position  $p$  is as follows:

Rewriting on the term part is as usual, we have to say how the labels of the new term look like.

Let  $\sigma l = t/p$ ,  $t' = t[p \rightarrow \sigma r]$  for some substitution  $\sigma$ .

We make  $t'$  a labelled term with labelling  $m'$  as follows:

If  $q > p$  or  $q \perp p$ , then  $m'(q) := m(q)$ .

Let  $x \in \text{Var}(r)$  and  $p_x$  be its position in  $l$ .

Let  $q_{x,i}$  be any position of  $x$  in  $r$ . Then  $t'/p.q_{x,i}$  has the same labelling as  $t/p.p_x$  with one exception:

If  $r = x$ , i.e., we have the application of a collapsing rule, then there may be two sources of a label for  $t'/p$ : the label of  $t/p$  and the label of  $t/p.p_x$ .

If neither  $t/p$  nor  $t/p.p_x$  are labelled, then  $t'/p$  has no label, too.

If exactly one of the terms  $t/p$  or  $t/p.p_x$  is labelled, then the new label of  $t'/p$  will be this one.

If both of them are labelled, i.e., we have a conflict, then there are two cases: in the case that  $t'/p$  is not an alien term, we delete both labels; otherwise, i.e.,  $t'/p.p_x$  is a variable, we use the label of  $t/p$ .  $\blacklozenge$

**2.3 Example.** Let  $g, b$  be 1-symbols and let  $F, A$  be 2-symbols. Let  $F(A, x) \rightarrow x$  be a rule. We give two examples of rewriting of labelled terms.

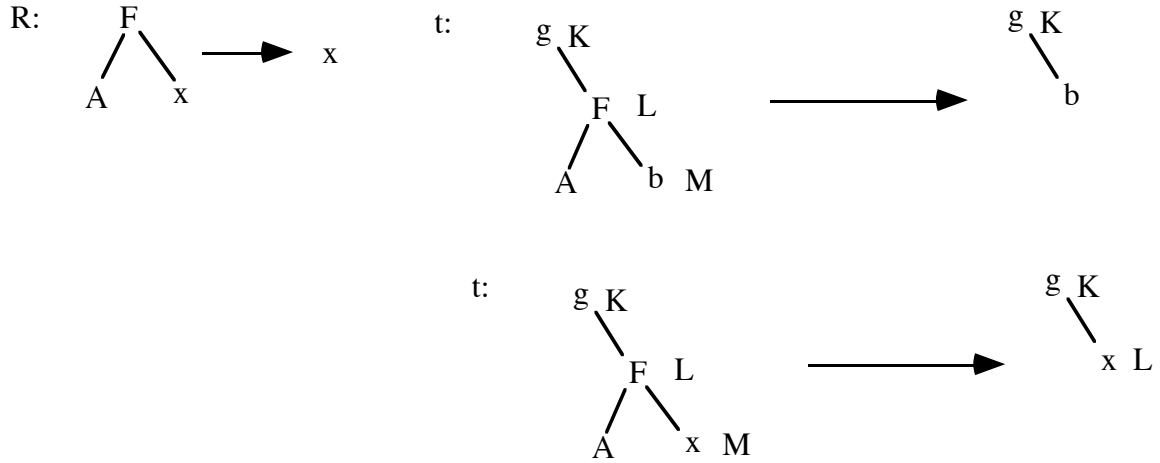


Figure 1

### 3. Reductions in $R_1 \oplus R_2$

Now we consider disjoint and left-linear term rewriting systems  $R_1$  and  $R_2$ , and study  $R_1 \sqcap \oplus \sqcap R_2$ -reductions of terms in  $\text{Term}(F_1 \cup F_2, V)$ .

#### 3.1 Examples.

i) This example demonstrates that it is not obvious how to construct a nonterminating reduction in one term rewriting system given a nonterminating reduction in the combination. However, it shows that the collapsing layers may play a rôle in such a construction process.

Let  $R_1: F(x, A) \rightarrow x, F(A, x) \rightarrow x,$

$R_2: h(g_1(x), g_2(y), z) \rightarrow h(z, z, z),$

$g_1(x) \rightarrow x$

$g_2(x) \rightarrow x.$

The following holds:  $R_1$  and  $R_2$  are left-linear and consistent. The latter holds, since the model  $\{1, 2\}$  with  $g_1(1) = g_2(1) = 1, g_1(2) = g_2(2) = 2, h(x, y, z) = z$  for all  $x, y, z \in \{1, 2\}$  is a model of  $=_{R_2}$ .

Let  $s = F(g_1(A), g_2(A))$  and  $t = h(s, s, s)$ . Then we have the cyclic reduction  $h(s, s, s) \xrightarrow{*} h(g_1(A), g_2(A), s) \xrightarrow{*} h(s, s, s)$ . To try some kind of projection in order to construct a nonterminating reduction sequence for  $R_2$  does not work: A corresponding term would be as follows:  $h(g_i(x), g_j(y), g_k(z))$ . However, the reduction  $\rightarrow_{R_2}$  terminates on such a term. Nevertheless,  $R_2$  is nonterminating:  $h(g_1(g_2(x)), g_1(g_2(x)), g_1(g_2(x))) \xrightarrow{*} h(g_1(x), g_2(x), g_1(g_2(x))) \xrightarrow{*} h(g_1(g_2(x)), g_1(g_2(x)), g_1(g_2(x)))$ .

ii) It is not true that consistency of a term rewriting system implies a unique collapsing of terms to some of their maximal alien subterms:

Consider  $R_1: f(x, a) \rightarrow x, f(b, x) \rightarrow x,$

$R_2: H(x) \rightarrow x$

Then  $R_1$  and  $R_2$  are consistent, but  $f(H(a), H(b))$  can be reduced to  $H(a)$  as well as to  $H(b)$ .

iii) A first idea to deal with collapsable layers is to consider the relation that is the transitive closure of all layer collapsings. The task is then to find out which of some deep alien

subterms may pop up to the surface. Given an infinite reduction one has to use a projection technique to produce an infinite reduction w.r.t a singel theory.

However, this does only work if some rewrite steps from both theories are considered:

Let  $R_1 := \{H_1(x) \rightarrow H_2(x,x), H_2(G_1(x), G_2(y)) \rightarrow y, G_1(x) \rightarrow x\}$

$R_2 := \{f(x) \rightarrow x, f(k(x)) \rightarrow x\}$  with  $F_2 = \{f,k,d\}$ .

Consider the term  $H_1(f(G_1(k(G_2(d(x))))))$ .

The following reduction shows, that  $d(x)$  may pop up at the top level.

$H_1(f(G_1(k(G_2(d(x)))))) \rightarrow H_2(f(G_1(k(G_2(d(x))))), f(G_1(k(G_2(d(x)))))) \xrightarrow{*}$

$H_2(G_1(k(G_2(d(x))))), f(k(G_2(d(x)))) \rightarrow H_2(G_1(k(G_2(d(x))))), G_2(d(x)) \rightarrow d(x)$ .

However, there is no reduction to this term, if we use layer by layer removal: Either

$H_1(f(G_1(k(G_2(d(x)))))) \rightarrow H_1(f(k(G_2(d(x)))) \rightarrow H_1(G_2(d(x)))$  or

$H_1(f(G_1(k(G_2(d(x)))))) \rightarrow H_1(G_1(k(G_2(d(x)))) \rightarrow H_1(k(G_2(d(x))))$ .  $\blacklozenge$

In the following we will use positions as labels, where the source of these positions is always the first term of a considered reduction.

**3.2 Definition.** Let  $R_i, i=1,2$  be disjoint, left-linear term rewriting systems and let  $t$  be a labelled term, such that the inital labelling is as follows: The root and every node of an alien position or of a variable is explicetely labelled with its position in  $t$ . Note that this labelling is injective.

The pair  $(p,q)$  is called an  **$m^*$ -collapsing**, if there is some reduction  $t \square^* \rightarrow t'$  and during reduction there occurs a conflict between label  $p$  and a label  $q$  with  $q < p$  where  $q$  is the position of a maximal alien subterm in  $t/p$ . The position  $p$  is called a modular collapsing position ( **$m^*$ -collapsing position**).

We call an  $m^*$ -collapsing  $(p,q)$  maximal, if the theory-length of  $p$  is maximal.

We use as a measure  $\mu$  of a term  $t$  the multiset of the theory-lengths of all  $m^*$ -collapsing positions in  $t$ . The usual multiset ordering is used for comparisons.  $\blacklozenge$

Observe that our labelling technique guarantees that a labelling is an injective mapping on the term starting a reduction. However, this property may be violated after some reductions, since rules like  $f(x) \rightarrow g(x,x)$  may generate several subterms with the same label. Note that there is a difference between  $m$ -collapsings in [Ma94a] and the  $m^*$ -collapsings defined here.

**3.3 Lemma.** Let  $R_i, i = 1,2$  be disjoint, left-linear and  $r$ -consistent.

Let  $p$  be a maximal  $m^*$ -collapsing position in  $t$  and  $(p,q_1), (p,q_2)$  be some  $m^*$ -collapsings. Then  $q_1 = q_2$ , i.e. maximal  $m^*$ -collapsings are unique.

**Proof.** Let  $(p,q_1)$  and  $(p,q_2)$  be two  $m^*$ -collapsings. Since  $p$  is a maximal one, there is no further collapsing in some alien subterm of  $t/p$ . We write  $t$  as a context  $t[[s_1, \dots, s_n]]$ , where  $s_1, \dots, s_n$  are the maximal alien subterms and the variables of  $t/p$ , which are at positions  $p$  with theory-length 0. Let  $p_i, i=1, \dots, n$  be the positions of the term  $s_i$  in  $t_0/p$ . Let  $i$  and  $j$  be the indices such that  $p_i = q_1$  and  $p_j = q_2$ . Let  $t[[x_1, \dots, x_n]]$  be a term, such that  $x_1, \dots, x_n$  are new variables. The two reductions that show that  $(p,q_1)$  and  $(p,q_2)$  are  $m^*$ -collapsings can be used to construct two different reductions  $t[[x_1, \dots, x_n]] \square^* \rightarrow x_i$  and  $t[[x_1, \dots, x_n]] \square^* \rightarrow x_j$  using the corresponding reductions in the top-layer of  $t/p$ . This is a contradiction to  $r$ -consistency of  $R_1$  or  $R_2$ .  $\blacklozenge$

In order to differentiate usual positions and positions that are used as labels, we will sometimes denote the latter by upper case letters.

**3.4 Definition.** Let  $R_i$ ,  $i = 1, 2$  be disjoint, left-linear and  $r$ -consistent. Let  $\text{red} = \{t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \dots\}$  be a reduction sequence w.r.t.  $R_1 \oplus R_2$ . Let the term  $t_0$  be labelled as in Definition 3.2 and let  $(P, Q)$  be a maximal  $m^*$ -collapsing. Then the transformation  $\tau$  w.r.t.  $(P, Q)$  is defined as follows:

We treat only the case where  $t_0/P$  is a 1-term, the case that  $t_0/P$  is a 2-term is then a dual case. Note that the position  $Q$  is unique by Lemma 3.3.

$\tau(t_i)$  is the term constructed from  $t_i$  for  $i = 1, 2, \dots$  by inserting  $t_0/P$  at every position  $q \neq \varepsilon$  in  $t_i$  that has the following properties:

- i)  $t_i/q$  is an alien 2-term
- ii) there is a position  $q' < q$ , such that  $q' \sqsubseteq$  is labelled with  $P$ .

The replacements are done as follows: Let  $q$  be such a position and let  $Q = P \bullet Q'$ . Then  $t_i/q$  is replaced by  $(t_0/P)[Q' \rightarrow t_i/q]$ , i.e., the term with root  $t_0/P$  and a hole at  $Q$  is plugged into the position  $q$ , where  $t_i/q$  is shifted down. Furthermore every term with label  $P$  is replaced by  $t_0/Q$ .

An illustration is given in Fig. 2. These replacements are done in parallel, but could also be done sequentially bottom-up. We can use  $\tau$  also for the transformations of positions: for positions  $r$  in  $t_i$  that are not below a node labelled  $P$ , we denote the image of  $r$  in  $\tau(t_i)$  by  $\tau_1(r)$ .

The case where  $P$  is the position of the root or some maximal alien subterm of  $t_0$  appears to be exceptional, since the  $m^*$ -collapsing layer is then completely removed, however, this case is also covered by our definitions.

Note that the illustration is an idealized one, in particular, there may be several alien terms labelled with  $P$ , hence the correct picture would be a tree-like tower instead of a linear tower.

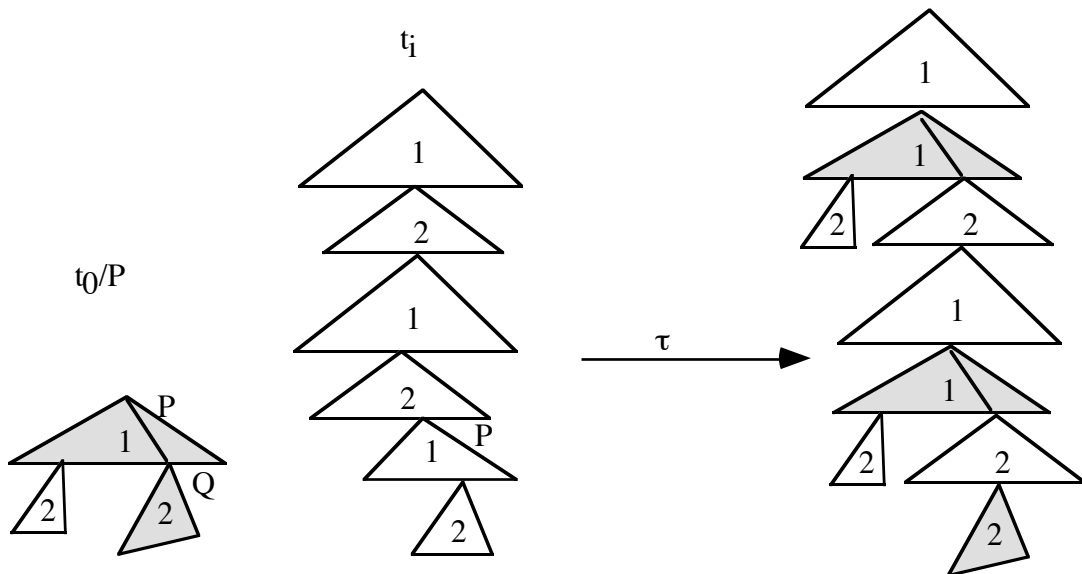


Figure 2

**3.5 Lemma.** Let  $R_i, i = 1,2$  be disjoint, left-linear and  $r$ -consistent. Let  $\text{red} = \{t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \dots\}$  be some reduction sequence w.r.t.  $R_1 \oplus R_2$ . Then

i)  $\tau(t_i) \square^* \rightarrow \tau(t_{i+1})$  for every  $i$ .

ii)  $\mu(\tau(t_0)) < \mu(t_0)$ , if  $\text{red}$  corresponds to  $\tau(t_0) \square^* \rightarrow \tau(t_1) \square^* \rightarrow \tau(t_2) \square^* \rightarrow \dots$

**Proof.**

i) We assume that the term  $t_0$  is labelled as in Definition 3.2. Let  $(P, Q)$  be some maximal  $m^*$ -collapsing. By symmetry, we can assume that the subterm in  $t_0$  labelled  $P$  is a 1-term.

We have to show that there is a reduction from  $\tau(t_i) \square^* \rightarrow \tau(t_{i+1})$  for every  $i$ . Assume the reduction  $t_i \rightarrow t_{i+1}$  is at position  $p$ . Then there are several cases:

- 1) In  $t_i$  there is a node with label  $P$  above  $p$ . In this case we have  $\tau(t_i) = \tau(t_{i+1})$ .
- 2) There is no node labelled  $P$  above  $p$  nor below  $p$ . In this case, the position  $\tau_i(p)$  in  $\tau(t_i)$  is defined, and we have  $\tau(t_i) \rightarrow \tau(t_{i+1})$  at position  $\tau_i(p)$  by the same reduction as  $t_i \rightarrow t_{i+1}$ .
- 3) There is a node labelled  $P$  properly below  $p$ . Then there are again several cases, which have to be considered in detail.
  - 3a) The reduction does not collapse a layer in  $t_i$ . The same reduction is then performed on  $\tau(t_i)$  at position  $\tau_i(p)$  resulting in  $r_i$ . The effect of the reduction in  $t_i$  may be to delete some nodes labelled  $P$ . Hence in order to reduce the possible differences between  $r_i$  and  $\tau(t_{i+1})$  some of the copies of  $t_0/P$  have to be removed. It is obvious that then  $\tau(t_{i+1})$  can be reached from  $r_i$ .
  - 3b) A 1-layer has been collapsed: The same argumentation as in 3a) is applicable.
  - 3c) A 2-layer has been collapsed that is not directly above the node labelled  $P$ :  
Again we can apply the argumentation of 3a)
  - 3d) A 2-layer has been collapsed that is directly above the node labelled  $P$ :  
Then again we have first to perform the same collapsing of a 2-layer in  $\tau(t_i)$  and then some removals of the plugged-in (partial) copies of  $t_0/P$  may be necessary. Furthermore, we have to make up the reductions in the image of the term that was labelled  $P$ . This, however, is possible, since the term in the image is exactly  $t_0/P$  and the term in  $t_i$  that was labelled  $P$  is a reduct of  $t_0/P$ .

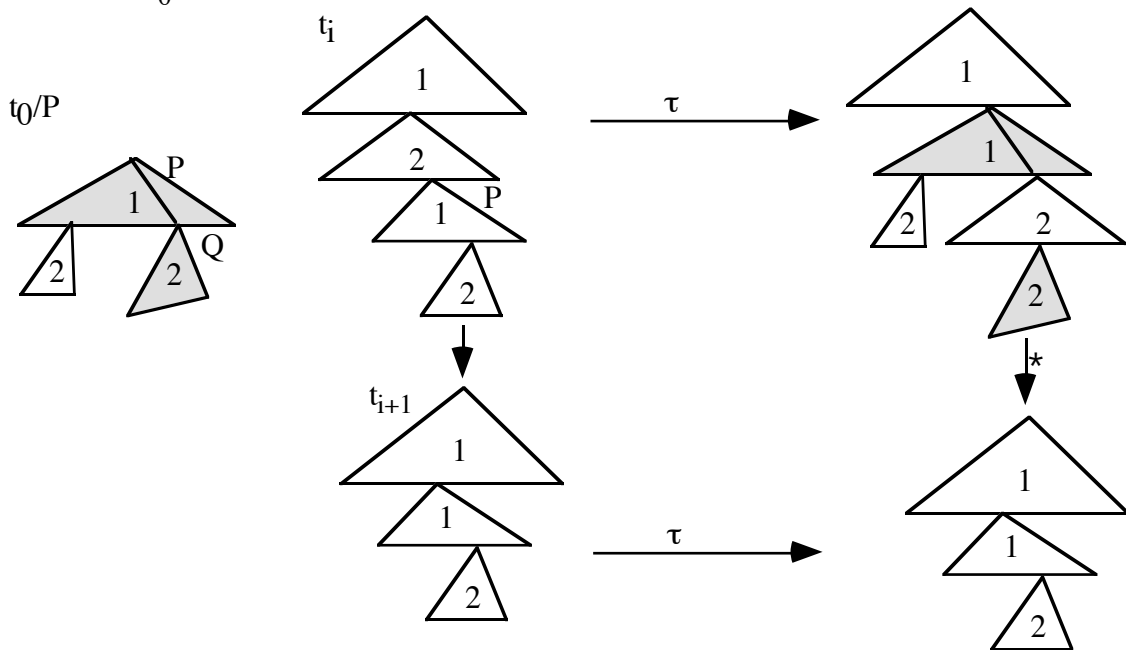


Figure 3



ii) Obviously, the transformation  $\tau$  reduces a maximal  $m^*$ -collapsing, namely  $t_0/P$ . Since there are no  $m^*$ -collapsings below  $P$ , the transformation may only introduce new collapsings which have a strictly smaller theory-length than  $P$ . Furthermore, the number of  $m^*$ -collapsing positions with the same theory-length as  $P$  is unchanged, since the transformation  $\tau$  does not influence the criterion for  $m^*$ -collapsings of these positions using i) of this lemma. This means,  $\mu(t_0) > \mu(\tau(t_0))$  with respect to the multiset-ordering.  $\blacklozenge$

**3.6 Lemma.** Let  $R_i$ ,  $i = 1, 2$  be disjoint, left-linear,  $r$ -consistent and terminating.

Then  $R_1 \oplus R_2$  is also terminating.

**Proof.** Assume by contradiction that there is some infinite reduction  $\text{red} = \{t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \dots\}$ . Furthermore, we can assume, that  $\text{red}$  is minimal among all such sequences, where we compare reduction sequences first by the rank of  $t_0$ , and second by  $\mu(t_0)$ .

1) The root is not an  $m^*$ -collapsing position. Furthermore there is an infinite number of reductions in the top layer:

If the root is an  $m^*$ -collapsing position, then there must be an infinite reduction sequence after the collapsing. Omitting the beginning of the reduction sequence, we get an infinite reduction sequence that starts with a term that has strictly smaller rank than  $t_0$ .

Now assume, the number of reductions in the top layer is finite. Then there is some index  $k$ , such that for all  $i \geq k$ , there are no more reductions in the top layer. But then there must be at least one maximal alien subterm  $s$  of  $t_k$ , from which an infinite reduction sequence starts. This term  $s$  has a strictly smaller rank than  $t_0$ , which is a contradiction to minimality of  $t_0$ .

2)  $\mu(t_0) = \emptyset$ : Assume  $\mu(t_0) \neq \emptyset$ . Then there exists some maximal  $m^*$ -collapsing in  $t_0$ . Instead of  $\text{red} = \{t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \dots\}$ , we consider the sequence  $\{\tau(t_0) \square^* \rightarrow \tau(t_1) \square \square^* \rightarrow \square \tau(t_2) \square^* \rightarrow \dots\}$ . Lemma 3.5 shows that this sequence is indeed a reduction sequence and that furthermore the new sequence is a strictly smaller one. By 1) there must be infinitely many reductions in the top layer. Hence we have a strictly smaller counterexample. This contradicts our assumption that  $\mu(t_0) \neq \emptyset$ .

3) Final contradiction. We have  $\mu(t_0) = \emptyset$ . This means there are no  $m^*$ -collapsings at all in  $t_0$ . Let  $s_0 \llbracket s_1, \dots, s_n \rrbracket = t_0$ , where  $s_0 \llbracket \dots, \dots \rrbracket$  denotes the top layer of  $t_0$ . By 1) there are infinitely many reductions in the top layer, hence there is an infinite reduction starting with the term  $s_0 \llbracket x_1, \dots, x_n \rrbracket$ , where  $x_i$  are new variables. This is an infinite reduction sequence in one of the  $R_i$ , which contradicts the assumption that  $R_i$  are both terminating.  $\blacklozenge$

Note that Lemma 3.6 provides a constructive method to obtain an infinite reduction for pure terms given an infinite reduction of mixed terms.

**3.7 Corollary.** Termination is modular for left-linear and  $r$ -consistent term rewriting systems.  $\blacklozenge$

**3.8 Example.** Let us revisit Example 3.1 above. We show how it may be transformed using the construction from Corollary 3.7.

Let  $R_1 : F(x, B) \rightarrow x, F(A, x) \rightarrow x,$

$$\begin{aligned}
R_2: \quad & h(g_1(x), g_2(y), z) \rightarrow h(z,z,z), \\
& g_1(x) \rightarrow x \\
& g_2(x) \rightarrow x.
\end{aligned}$$

Let  $s = F(g_1(A), g_2(B))$  and  $t = h(s,s,s)$ . Then we have the cyclic reduction  $h(s,s,s) \xrightarrow{*} h(g_1(A), g_2(B), s) \xrightarrow{*} h(s,s,s)$ . The construction now means to apply the transformation  $\tau$  several times. There are two maximal  $m^*$ -collapsing positions, namely the terms  $g_1(A)$  and  $g_2(B)$ . Selecting one for the transformation, we first obtain the term  $h(s', s', s')$  with  $s' = g_1(F(A, g_2(B)))$ . The reduction sequence is now  $h(s', s', s') \rightarrow h(g_1(F(A,B)), s', s') \rightarrow h(g_1(A), \square s', s') \rightarrow h(g_1(A), F(A, g_2(B)), s') \rightarrow h(g_1(A), \square g_2(B), s') \rightarrow h(s', s', s')$ . The next operation would construct the term  $h(s'', s'', s'')$  with  $s'' = g_1(g_2(F(A, B)))$ . Now there is an obvious infinite reduction using only 1-terms and 1-rules:  $h(s'', s'', s'') \rightarrow h(g_1(F(A, B)), s'', s'') \rightarrow h(g_1(F(A, B)), g_2(F(A, B)), s'') \rightarrow h(s'', s'', s'')$ . Replacing  $F(A,B)$  by some variable  $x$ , we have constructed a cyclic reduction of a 1-term:  $h(g_1(g_2(x)), g_1(g_2(x)), g_1(g_2(x))) \rightarrow h(g_1(x), g_1(g_2(x)), g_1(g_2(x))) \rightarrow h(g_1(x), g_2(x), g_1(g_2(x))) \rightarrow h(g_1(g_2(x)), g_1(g_2(x)), g_1(g_2(x)))$ . This reduction is the same as we have guessed in Example 3.1.i).  $\blacklozenge$

The theorem of Toyama, Klop, and Barendregt on the modularity of termination for confluent and left-linear term rewriting systems can be obtained as a corollary of Corollary 3.7.

**3.9 Corollary.** Termination is modular for left-linear and confluent term rewriting systems.

**Proof.** This follows since confluent term rewriting systems are also r-consistent.  $\blacklozenge$

The above result in Corollary 3.7 allows us to prove the following modularity result, thus solving an open problem posed by K. Rao in [Ra93]:

**3.10 Corollary.** Termination is modular for left-linear and non-erasing term rewriting systems.

**Proof.** This follows since non-erasing term rewriting systems are also r-consistent.  $\blacklozenge$

This is in contrast to the fact that termination is not modular for non-erasing term rewriting systems as proved in [Oh94].

The following proposition clarifies the structure of counterexamples to the modularity of termination in the case of left-linear term rewriting systems.

**3.11 Proposition.** Let  $R_1$  and  $R_2$  be two disjoint, left-linear and terminating term rewriting systems, and let  $R_1 \oplus R_2$  be non-terminating. Then one of the term rewriting systems  $R_i$  is r-consistent, while the other one is not r-consistent

**Proof.** Let  $\text{red} = \{t_0 \rightarrow t_1 \rightarrow \dots\}$  be an infinite reduction w.r.t.  $R_1 \oplus R_2$ . We can choose  $\text{red}$ , such that the rank of  $t_0$  is minimal. Then the rank of  $t_i$  is constant and the number of reductions in the top layer is infinite. Without loss of generality we can assume that  $\text{Th}(t_0) = 1$ .

Suppose  $R_1$  is not r-consistent. Then there is an  $R_1$ -term  $s$ , such that  $s \square^* \rightarrow x$  and  $s \square^* \rightarrow y$  for different variables  $x, y$ . It follows immediately that there is a sequence of terms  $s_i \llbracket x_1, \dots, x_n \rrbracket$ , such that  $s_i \llbracket x_1, \dots, x_n \rrbracket \square^* \rightarrow x_j$  for every  $j = 1, \dots, i$ . Now we construct a

reduction sequence in  $R_1$  as follows:  $\alpha(t_i)$  is defined as the term, where every alien 2-term  $s$  is replaced as follows. If  $s$  has  $n$  maximal alien 1-subterms  $r_1, \dots, r_n$  then it is replaced by  $s_n[[r_1, \dots, r_n]]$ . This transformation can be done in parallel or bottom-up. The term  $\alpha(t_i)$  is then a 1-term for all  $t_i$ . Using the reductions in  $\text{red}$  we show that  $\alpha(t_i) \rightarrow \alpha(t_{i+1})$ . If  $t_i \rightarrow t_{i+1}$  is a reduction using a 2-rule, then there are two possibilities: If an alien 2-term  $r$  is replaced by one of its alien 1-terms  $r_k$ , we use the reduction  $s_n[[r_1, \dots, r_n]] \square^* \rightarrow r_k$  to reduce  $\alpha(r)$  to  $r_k$ ; In the other case we simply ignore the reduction. If it is a reduction using a 1-rule, then we make the corresponding reduction in  $\alpha(t_i)$ , which yields  $\alpha(t_{i+1})$ . Since the number of reductions in the top layer is infinite, we get an infinite reduction of a 1-term, which contradicts the assumption that  $R_1$  is terminating. Thus  $R_1$  is  $r$ -consistent. By Corollary 3.7,  $R_2$  cannot be  $r$ -consistent.  $\blacklozenge$

The following Theorem 3.12 shows that  $r$ -consistency is a modular property of left-linear term rewriting systems. This permits to use the result of Corollary 3.7 also for a disjoint union of several  $r$ -consistent term rewriting systems.

**3.12 Theorem.**  $r$ -consistency is a modular property of left-linear term rewriting systems.

**Proof.** Let  $R_i$ ,  $i= 1,2$ , be disjoint, left-linear and  $r$ -consistent term rewriting systems. Assume there is a term  $t_0$ , such that there are two reductions  $t_0 \square^* \rightarrow x$  and  $t_0 \square^* \rightarrow y$  and furthermore assume that  $\mu(t_0)$  is minimal. Since  $t_0$  collapses to a variable, there must be some  $m^*$ -collapsing in  $t_0$ . Applying the transformation  $\tau$ , we get a further term  $\tau(t_0)$  that is reducible to  $\tau(x) = x$  and  $\tau(y) = y$ . Furthermore  $\mu(\tau(t_0)) < t_0$ . By minimality,  $t_0$  must be a 1-term or 2-term, which contradicts the assumption that both  $R_i$  are  $r$ -consistent.  $\blacklozenge$

The proof technique used in this paper appears not to be extendable to the sum of term rewriting systems where there may be common constructors, i.e., function symbols that do not appear at the top in the left hand sides of rules (see [KO91, MT91, Ohl94]). One argument is that the deletion part of our argument does not work any more, since the plug-in operation in Definition 3.4 may copy constructors. Furthermore there are simple counterexamples to the theorem, if common constructors are permitted, even in the restricted case of constructor-based systems (i.e., symbols not in top position in the left hand sides must be constructors):

Let  $R_1 : h(a,b,x) \rightarrow h(x,x,x)$   
 $R_2 : C \rightarrow b, C \rightarrow a$

Then  $R_1$  and  $R_2$  are left-linear, terminating,  $r$ -consistent, non-erasing and collapse-free. The common constructors are the constants  $a$  and  $b$ . However, there is an infinite reduction in the sum  $R_1 \oplus R_2$ :

$h(C,C,C) \rightarrow h(a, C,C) \rightarrow h(a, b, C) \rightarrow h(C,C,C).$

## 4. Conclusion.

This result generalizes the theorem from [TKB89] stating that termination is a modular property of confluent and left-linear term rewriting systems. Our proof technique is different from theirs, but much simpler.

Our result extends the range of direct sums of term rewriting systems, where termination can be concluded from the termination of the constituents. Consider for example the TRS:  $R = \{f(x, f(y,z)) \rightarrow f(x, g(f(y, x))), g(h(x)) \rightarrow x, h(x) \rightarrow x\}$ . This term rewriting system is terminating, but not confluent. If we try to make a completion, compute a critical pair  $(g(x), x)$ , and add the rule  $g(x) \rightarrow x$ , then the system will be non longer terminating, since then the term  $f(x, f(x, x))$  starts a cyclic reduction. Hence the theorem of [TKB89] cannot be applied to this example.

## References:

- [BD86] L. Bachmair, N. Dershowitz, Commutation, transformation and termination, Proc 8th Conference on Automated Deduction, Springer LNCS 230, pp. 5-20, (1986)
- [De87] N. Dershowitz, Termination of rewriting, Journal of Symbolic Computation **3**(1), pp. 69-116, (1987)
- [DJ90] N. Dershowitz, J.-P. Jouannaud, Rewrite systems, in J. van Leeuwen (ed.) Handbook of theoretical computer science: Formal models and semantics, Vol B, pp. 243-320, (1990)
- [Dr89], K. Drosten, Termersetzungssysteme, Informatik-Fachberichte 210, Springer-Verlag, (1989), in German
- [Gr92] B. Gramlich, Relating innermost, weak, uniform and modular termination of term rewriting systems, Int. Conf. on Logic Programming and Automated Reasoning, Springer Lecture Notes in Computer Science 624, pp.285- 296, (1992)
- [Hu80] G.Huet, Confluent reductions: abstract properties and applications to term rewriting systems, Journal of the ACM **27**(4), pp. 797-821, (1980)
- [Kl92] J.W.Klop, Term rewriting systems, in Vol 2 of Handbook of Logic in Computer Science (eds. S. Abramsky, Dov M. Gabbay , T.S.E. Maibaum), pp. 1-116, (1992)
- [KO91] M. Kurihara and A. Ohuchi, Modularity of simple termination in term rewriting systems, Journal of Information Processing **14**(3), IPS of Japan, pp. 357-358, (1991)
- [Mi89] A. Middeldorp, A sufficient condition for the termination of the direct sum of term rewriting systems, Proc. 4th IEEE Symposium on Logic in Computer Science, pp. 396-401, (1989)
- [Ma94a] M. Marchiori, Modularity of  $UN \rightarrow$  for left-linear term rewriting systems, technical report CS-R9433, CWI, Amsterdam, (1994)
- [Ma94b] M. Marchiori, Modularity of completeness revisited, technical report 6, Department of Pure and Applied Mathematics, University of Padova, Italy, (1994)
- [Mi90] A. Middeldorp, Modular properties of term rewriting systems, Ph.D.thesis, Vrije Universiteit, Amsterdam, (1990)
- [MT91] A. Middeldorp, Y. Toyama, Completeness of combinations of constructor systems, Proc. 4th RTA-91, LNCS 488, pp. 188- 199, (1991)
- [Oh93] E. Ohlebusch, Termination is not modular for confluent variable-preserving term rewriting systems, draft , Universität Bielefeld, (1993)
- [Oh94] E. Ohlebusch, Modular properties of composable term rewriting systems, PhD thesis, Universität Bielefeld, (1994)
- [Ra93] K. Rao, Termination Characteristics of Logic Programs, PhD Thesis, Tata Institute of Fundamental Research, Bombay, (1993).

- [Ru87] M. Rusinowitch: On termination of the direct sum of term rewriting systems, Information processing letters **26**, pp. 65-70, (1987)
- [SS89] M. Schmidt-Schauß, Unification in a Combination of Arbitrary Disjoint Equational Theories, J. Symbolic Computation **8**, pp. 51-99, (1989)
- [TKB89] Y. Toyama, J.W.Klop, H.P.Barendregt, Termination for the direct sum of left-linear term rewriting systems, Proc. 3rd International Conference on Rewriting Techniques and applications, Springer LNCS 355, pp. 477-491, (1989)
- [To87a] Y. Toyama, Counterexamples to the Termination for the direct sum of term rewriting systems, Information processing letters **25**, pp. 141-143, (1987)
- [To87b] Y. Toyama, On the Church-Rosser Property for the direct sum of term rewriting systems, Journal of the ACM **34** (1), pp. 128-143, (1987)