Modularity of Completeness Revisited

Massimo Marchiori

Department of Pure and Applied Mathematics
University of Padova
Via Belzoni 7, 35131 Padova, Italy
`max@hilbert.math.unipd.it`

Abstract. One of the key results in the field of modularity for Term Rewriting Systems is the modularity of completeness for left-linear TRSs established by Toyama, Klop and Barendregt in [TKB89]. The proof, however, is quite long and involved. In this paper, a new proof of this basic result is given which is both short and easy, employing the powerful technique of 'pile and delete' already used with success in proving the modularity of $UN^{\rightarrow}$. Moreover, the same proof is shown to *extend* the result in [TKB89] proving modularity of termination for left-linear and consistent with respect to reduction TRSs.

1 Introduction

The property of completeness for a Term Rewriting System, that is being Church-Rosser and terminating, is of fundamental importance in every application of rewriting. One of the cornerstones as far as completeness is concerned is the smart result obtained by Toyama, Klop and Barendregt in [TKB89] (see also [TKB94]) asserting the modularity of completeness for left-linear TRSs (a property is modular provided it is valid for two TRSs if and only if it holds for their disjoint sum). However, as stated by the same authors in their paper, the proof used to obtain such result is 'rather intricate and not easily digested'.

In this paper we show that the same result can be obtained making use of the technique of 'pile and delete' developed in [Mar93] to prove the modularity of $UN^{\rightarrow}$ (uniqueness of normal form with respect to reduction) for left-linear TRSs. The proof obtained is extremely short and rather intuitive, thus providing new insights in the study of completeness.

The established link between proof methods of completeness and $UN^{\rightarrow}$ should not come as a total surprise, however, since it is easily seen that:

$$\text{Completeness} = \text{Church-Rosser} + \text{Termination} = UN^{\rightarrow} + \text{Termination}.$$

In fact, we will see that the given proof even *extends* the result in [TKB89], proving modularity of termination for left-linear and *consistent with respect to reduction* TRSs (a TRS is consistent with respect to reduction if a term cannot be rewritten to two different variables).

The paper is organized in the following way: Section 2 gives the necessary preliminaries, Section 3 states the main theorem about modularity of completeness for left-linear TRSs, and finally Section 4 shows how exactly the same proof yields the stronger result of modularity of termination for left-linear and consistent with respect to reduction TRSs.

2 Preliminaries

We assume the reader to be familiar with the basic notions regarding Term Rewriting Systems: the notation used is essentially the one in [Klo92] and [Mid90].

The properties of being confluent (Church-Rosser) and terminating (Strongly Normalizing) will be indicated respectively with CR and SN. Contexts will be denoted as usual with $\Box$ and with square brackets ($C[\cdots]$). Throughout the paper we will indicate with $\mathcal{A}$ and $\mathcal{B}$ the two TRSs to operate on, their corresponding sets of function symbols with $\mathcal{F}_{\mathcal{A}}, \mathcal{F}_{\mathcal{B}}$, the variables set as $\mathcal{V}$, and the set of terms built from some set of function symbols $\mathcal{F}$ and $\mathcal{V}$ as $\mathcal{T}(\mathcal{F})$.

When not otherwise specified, all symbols and notions not having a TRS label are to be intended operating on the disjoint sum $\mathcal{A} \oplus \mathcal{B}$. For better readability, we will talk of function symbols belonging to $\mathcal{A}$ and $\mathcal{B}$ like *white* and *black* functions, indicating the first ones with upper case functions, and the second ones with lower case. Variables, instead, have no colour.

Definition 2.1 The *root* symbol of a term $t \in \mathcal{T}(\mathcal{F}_{\mathcal{A}} \cup \mathcal{F}_{\mathcal{B}})$ is f provided $t = f(t_1, \ldots, t_n)$, and t itself otherwise. $\Box$

Let $t = C[t_1, \ldots, t_n] \in \mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2)$ and $C \neq \Box$; we write $t = C[\![t_1, \ldots, t_n]\!]$ if $C[\ldots]$ is an $\mathcal{F}_{\mathcal{A}}$-context and each of the t_i has $root(t_i) \in \mathcal{F}_{\mathcal{B}}$, or vice versa (exchanging $\mathcal{A}$ and $\mathcal{B}$). The *topmost homogeneous part* (briefly *top*) of a term $C[\![t_1, \ldots, t_n]\!]$ is the context $C[\![\cdots]\!]$.

Definition 2.2 The *rank* of a term $t \in \mathcal{T}(\mathcal{F}_{\mathcal{A}} \cup \mathcal{F}_{\mathcal{B}})$ is 1 if $t \in \mathcal{T}(\mathcal{F}_{\mathcal{A}})$ or $t \in \mathcal{T}(\mathcal{F}_{\mathcal{B}})$, and $\max_{i=1}^{n}\{rank(t_i)\} + 1$ if $t = C[\![t_1, \ldots, t_n]\!]$ $(n > 0)$. $\Box$

The following well known lemma will be implicitly used in the sequel:

Lemma 2.3 ([Toy87]) $s \twoheadrightarrow t \Rightarrow rank(s) \geq rank(t)$

Proof. Clear. $\Box$

Definition 2.4 The multiset $S(t)$ of the *special subterms* of a term t is

1. $S(t) = \begin{cases} \{t\} & \text{if } t \in (\mathcal{T}(\mathcal{F}_{\mathcal{A}}) \cup \mathcal{T}(\mathcal{F}_{\mathcal{B}})) \backslash \mathcal{V} \\ \emptyset & \text{if } t \in \mathcal{V} \end{cases}$
2. $S(t) = \cup_{i=1}^{n} S(t_i) \cup \{t\}$ if $t = C[\![t_1, \ldots, t_n]\!]$ $(n > 0)$ $\Box$

Note that this definition is slightly different from the usual ones in the literature (for example in [Mid90]), since here variables are not considered special subterms.

If $t = C[\![t_1, \ldots, t_n]\!]$, the t_i are called the *principal* special subterms of t. Furthermore, a reduction step of a term t is called *outer* if the rewrite rule isn't applied in the principal special subterms of t.

A (strict) *partial order* on the special subterms of a term can be naturally given defining $t_1 \succ t_2$ iff t_2 is a proper special subterm of t_1 (that is $t_1 = C[\![\ldots t_2 \ldots]\!]$).

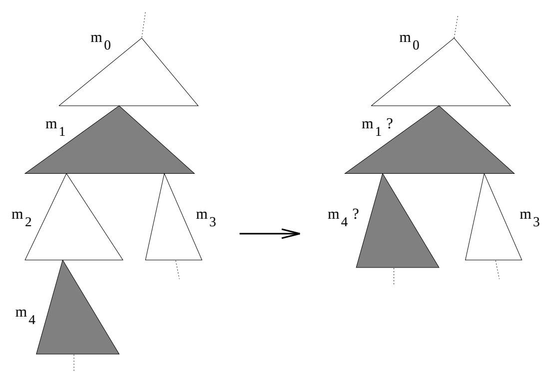
The following proposition will reveal useful:

Proposition 2.5 *If $\mathcal{A}$ and $\mathcal{B}$ are left-linear, then rewrite rules that have the possibility to act outer on a special subterm t are exactly those that have the possibility to act on its top.*

Proof. Let $t = C[\![t_1, \ldots, t_n]\!]$: since $t_1, \ldots, t_n$ have a root belonging to the other TRS (with respect to C), they are matched by variables from any rewrite rule applicable to C, and for the left-linearity assumption these variables are independent each other. □

2.1 Marking and Collapsing

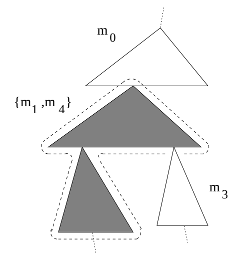
To be able to describe the special subterms of a given term throughout a reduction, it is natural to develop a concept of (modular) marking. A first, naïve approach of modular marking for a term is to take an assignment from the multiset of its special subterms to a (fixed) set of markers. So, for instance, given the term $F(f(G, a), H)$, we could mark $F(\Box, H)$ to m_1, $f(\Box, a)$ to m_2, G to m_3. Then reductions steps, as usual, should preserve the markers. However, this simple definition presents a problem, since for one case there is ambiguity: when a collapsing rule makes an *inner* top vanish. In this case, we have the following situation:



and we have a conflict between m_1 and m_4.

This situation is dealt with by defining a *modular marking* for a term to be an assignment from the multiset of its special subterms to *sets* of markers, and taking in the ambiguous case just described the union of the marker sets of the two special subterms involved.

Thus, the previous example would give (singletons like $\{m_3\}$ are written simply m_3):



When this situation occurs, we say that the special subterm m_4 has been *absorbed* by m_1, and the special subterm m_2 has had a *modular collapsing* (briefly *m-collapsing*).

When dealing with reductions $t \twoheadrightarrow t'$ we will always assume, in order to distinguish all the special subterms, that the initial modular marking of t is injective and maps special subterms to singletons.

Inside a reduction a notion of descendant for every special subterm can be defined: in a reduction a special subterm is a *descendant* (resp. *pure descendant*) of another if the set of markers of the former contains (resp. is equal to) the set of markers of the latter. Note, en passant, that due to the presence of duplicating rules, there may be more than one descendant, or even none (due to erasing rules). Observe also that, since in a reduction without m-collapsings all the descendants are pure, the first special subterm to m-collapse in a generic reduction is a pure descendant. Hence it readily holds the following:

Fact: *A reduction has m-collapsings iff a pure descendant m-collapses.*

One of the reasons cited in [TKB89] for the difficulty in treating the problem of the completeness modularity is the *nondeterministic* collapsing behaviour given by ambiguities among rewrite rules. The following result shows a case in which things behave nicely:

Proposition 2.6 *Let $\mathcal{A}$ be left-linear and $UN^{\rightarrow}$, and $t = C[\![t_1, \ldots, t_n]\!]$ a top white special subterm: if t m-collapses into t_i $(1 \leq i \leq n)$ via a white reduction (i.e. using only rules from $\mathcal{A}$), then the index i is unique.*

Proof. Since $\mathcal{A}$ is left-linear, by Proposition 2.5 the white reduction depends only on the top of t. Hence, if we take instead of $t = C[\![t_1, \ldots, t_n]\!]$ a term $t' = C[\![X_1, \ldots, X_n]\!]$ (with $X_1, \ldots, X_n$ new fresh variables), then every previous white reduction that m-collapsed t to t_i can be repeated on t' to reduce it to X_i, and if the index i were not unique t' could be reduced to different normal forms, contradicting the fact $\mathcal{A}$ is $\mathrm{UN}^{\rightarrow}$. □

3 The Main Theorem

We have arrived to the main theorem: the essence of the new proof is to use the pile and delete technique developed in [Mar93] to transform every infinite reduction into an infinite reduction without m-collapsings, and then prove this kind of reductions cannot exist.

Theorem 3.1 *Completeness is a modular property for left-linear TRSs.*

Proof. Since CR is a modular property ([Toy87]), it only remains to prove that if $\mathcal{A}$ and $\mathcal{B}$ are left-linear and complete, then $\mathcal{A} \oplus \mathcal{B}$ is SN (the reverse implication is trivial). So, suppose ab absurdo there exists an infinite reduction. Then take an infinite reduction $u \to u_1 \to u_2 \to \ldots$ that has the starting term u with *minimum rank* among the terms with an infinite reduction.

First, note how the number of reduction rules acting on the top of u are infinite: if it were not so, then we would have a finite number of proper special subterms, and since all of these are SN by the rank minimality of u no infinite reduction would be possible.

We will show that from this reduction a new infinite one can be obtained, which is without m-collapsings (employing the same 'pile and delete' technique as in [Mar93]).

If the reduction is already without m-collapsings, the assertion is trivially satisfied.

Thus, suppose in the reduction some m-collapsing is performed. Select a special subterm of u that has rank minimal amongst the ones with a pure descendant that m-collapses in the reduction itself: say $t = \tau[\![t_1, \ldots, t_n]\!]$. Because of its rank minimality, t must m-collapse by Proposition 2.6 into a fixed principal subterm, namely t_i. The top of t, say τ, might be absorbed from other tops of $(\prec)$-greater special subterms in the reduction. All of these special subterms are descendants of some special subterms of the start $r_1, \ldots, r_k$.

We now modify u via the 'pile and delete' process.

First, we 'pile' $\tau[\![t_1, \ldots, t_{i-1}, \Box, t_{i+1}, \ldots, t_n]\!]$ just below the tops of the $r_1, \ldots, r_k$, that is to say if $r_i = r_i[\![s_1, \ldots, s_v]\!]$ and t is in s_j (viz. $t \prec s_j$), then r_i is replaced with

$$r_i[\![s_1, \ldots, s_{j-1}, \tau[t_1, \ldots, t_{i-1}, s_j, t_{i+1}, \ldots, t_n], s_{j+1} \ldots, s_v]\!]$$

The situation is shown in Figure 1.

The intuition is that we inserted copies of t where needed later in the reduction for absorption. So now we can 'delete' it replacing t by t_i (see Figure 2).

How can we get an infinite reduction from this modified starting term? We simply use the previous infinite reduction, with the following modifications:

- We drop the rules from the original reduction acting on pure descendants of t but not on pure descendants of t_i.
- When a descendant of t was absorbed by, say, $\bar{r}_q$, we piled to its ancestor r_p (and so to its descendant $\bar{r}_q$) in that place $\tau[\![t_1, \ldots, t_{i-1}, \Box, t_{i+1}, \ldots, t_n]\!]$, whereas the old descendant of t is now the corresponding descendant of t_i, so it only remains to reduce the piled $\tau[\![t_1, \ldots, t_{i-1}, \Box, t_{i+1}, \ldots, t_n]\!]$ as previously in the reduction to obtain exactly the same situation as before, and the new reduction can proceed in the mimicking of the old reduction (see Figure 3). Note how these postponed reductions provoke no m-collapsings.
- We inserted $\tau[\![t_1, \ldots, t_{i-1}, \Box, t_{i+1}, \ldots, t_n]\!]$ below all the $r_1, \ldots, r_k$, but actually pure descendants of t may be absorbed in the old reduction only by part of

the descendants of these special subterms.

But we can get rid of these superfluous occurrences of material acting, as hinted previously, with the rules that in the initial reduction made $\tau[\![t_1, \ldots, t_{i-1}, \Box, t_{i+1}, \ldots, t_n]\!]$ collapse into $\Box$: they are applied to all of these extra descendants when the piled material is not needed any more. This means that these 'deleting sequences' must be applied

i) When in the sequel of the original reduction the descendant of an r_p will not absorb a pure descendant of t any more.

ii) When the descendant of an r_p absorbs another descendant of an r_q (Fig. 4).

Again, it is immediate to see these deleting sequences produce no m-collapsings.

This way we have obtained a new reduction that is infinite since all of the infinite rules acting on the top of the start term are still present. Note that left-linearity, via Proposition 2.5, was essential to be able to mimic the old reduction.

This new reduction has the number of special subterms of the start with a descendant that m-collapses in the reduction itself diminished by one: indeed, t is no more present, and as remarked no new m-collapsings are introduced modifying the original reductions.

Therefore, repeating this 'pile and delete' process leads, ultimately, to an infinite reduction without m-collapsings.

But this is impossible: this reduction has an infinite number of reduction rules acting on the top of the starting term, as seen before (alternatively, note this also follows from the fact the 'pile and delete' process does not increase the rank of the term), but since there are no m-collapsings the principal subterms of the starting term can be substituted by fresh variables without preventing the application of these infinite number of reduction rule (Proposition 2.5). Hence we obtain an infinite reduction of a term of an unique colour, contradicting the fact $\mathcal{A}$ and $\mathcal{B}$ are SN. $\Box$

4 Remarks

Note how the previous proof relies upon modularity of CR ([Toy87]) to show that only SN has to be proved: this is not necessary, however, since as noticed in the introduction CR + SN = $\text{UN}^{\rightarrow}$ + SN, and therefore the modularity of $\text{UN}^{\rightarrow}$ for left-linear TRSs proved in [Mar93] could be used as well, thus giving a proof completely based on the 'pile and delete' technique.

In fact, in [Mar93] it was noticed that the 'pile and delete' technique does not need the full power of $\text{UN}^{\rightarrow}$, but it can be applied under the weaker assumption of *consistency with respect to reduction* (briefly $\text{CON}^{\rightarrow}$), that is satisfied if a term cannot be rewritten to two different variables. This is true since the 'pile and delete' technique essentially relies upon Proposition 2.6, that still holds if $\text{CON}^{\rightarrow}$ is required in place of $\text{UN}^{\rightarrow}$. Hence, it was shown in [Mar93] that exactly the same proof used for the modularity of $\text{UN}^{\rightarrow}$ proved that $\text{CON}^{\rightarrow}$ is modular for left-linear TRSs.

The same observation is pertinent here: (the proof of) Theorem 3.1 *extends* the result in [TKB89] showing that

Theorem 4.1 *Termination is a modular property for left-linear and consistent with respect to reduction TRSs.*

Observe that, using the aforementioned modularity of $\mathrm{CON}^{\rightarrow}$, it even holds the stronger result that $\mathrm{SN} + \mathrm{CON}^{\rightarrow}$ is a modular property for left-linear TRSs.

An easy consequence of the above theorem is the following. Recall that a TRS is said *non-erasing* if for every its rule $l \rightarrow r$ the variables in r are the same as in l. It is immediate to see that if a TRS is non-erasing then it is consistent w.r.t. reduction. Hence by Theorem 4.1 we obtain right away

Corollary 4.2 *Termination is a modular property for left-linear and non-erasing TRSs.*

Acknowledgments: Thanks are due to Jan Willem Klop for his continuous support and encouragement in getting a better and shorter proof. Moreover, I want to thank Aart Middeldorp for careful reading an earlier version of this paper and for having implicitly suggested me that the proof here given was not only a great simplification of [TKB89] but also an extension: indeed, I heard from him about the existence of the draft [SSP94], and that title gave me the suggestion to weaken $\mathrm{UN}^{\rightarrow}$ in $\mathrm{CON}^{\rightarrow}$ (incidentally, we note the basic character of the pile and delete technique developed in [Mar93], since in [SSP94] an analogous technique is employed). Thanks also to Vincent van Oostrom and Enno Ohlebusch.

References

[Klo92] J.W. Klop. Term rewriting systems. In S. Abramsky, Dov M. Gabbay, and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 2, chapter 1, pages 1–116. Clarendon Press, Oxford, 1992.

[Mar93] M. Marchiori. Modularity of $\mathrm{UN}^{\rightarrow}$ for left-linear term rewriting systems. Draft, March 1993. Extended and revised version: Technical Report CS-R9433, CWI, Amsterdam.

[Mid90] A. Middeldorp. *Modular Properties of Term Rewriting Systems.* PhD thesis, Vrije Universiteit, Amsterdam, November 1990.

[SSP94] M. Schmidt-Schauß and S. Panitz. Modular termination of consistent and left-linear TRSs. Draft, May 1994.

[TKB89] Y. Toyama, J.W. Klop, and H.P. Barendregt. Termination for the direct sum of left-linear term rewriting systems. In N. Dershowitz, editor, *Proceedings of the 3rd RTA*, vol. 355 of *LNCS*, pp. 477–491, Springer, 1989. Extended version: Technical Report CS-R8923, CWI, Amsterdam.

[TKB94] Y. Toyama, J.W. Klop, and H.P. Barendregt. Termination for direct sums of left-linear complete term rewriting systems. *JACM*, 1994. To appear.

[Toy87] Y. Toyama. On the Church-Rosser property for the direct sum of term rewriting systems. *JACM*, 1(34):128–143, 1987.

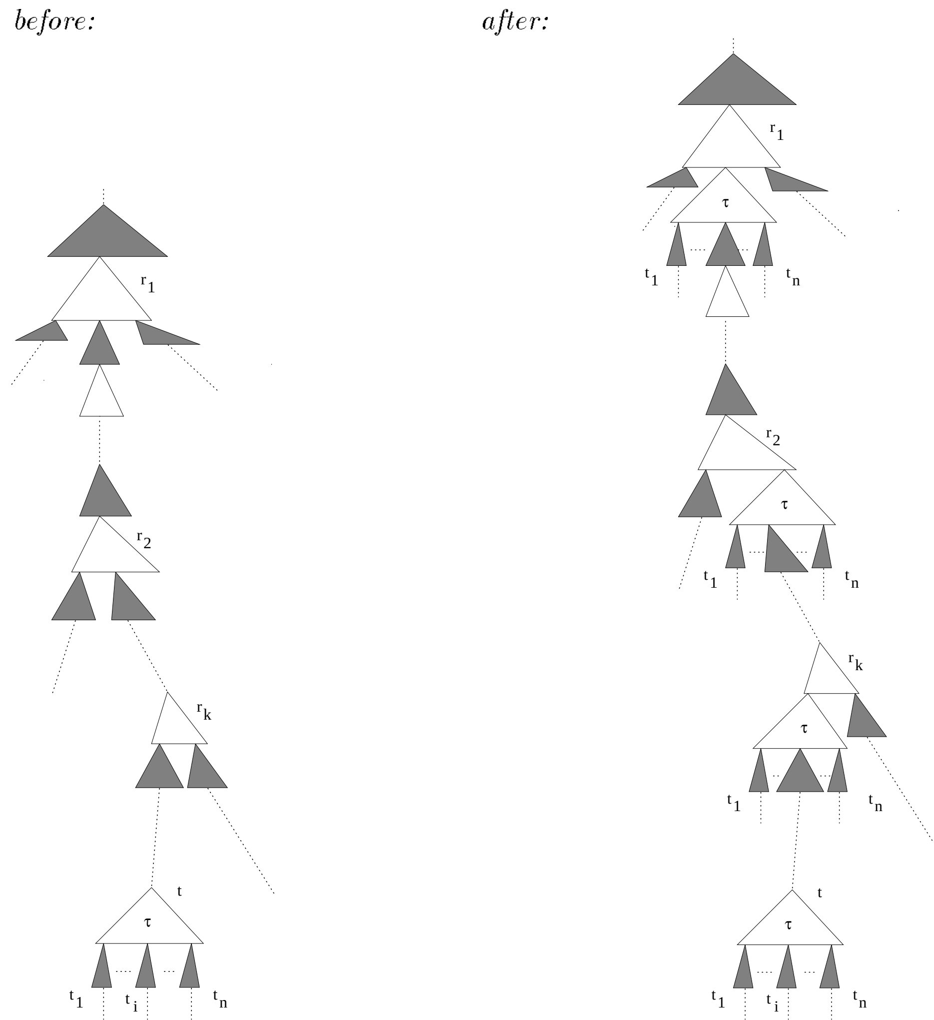


Fig. 1. The 'pile' process.

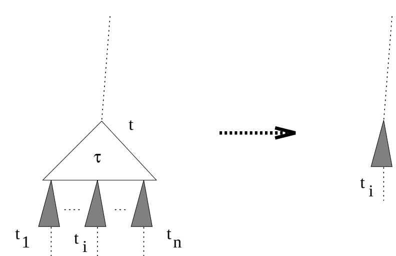


Fig. 2. The 'delete' process.

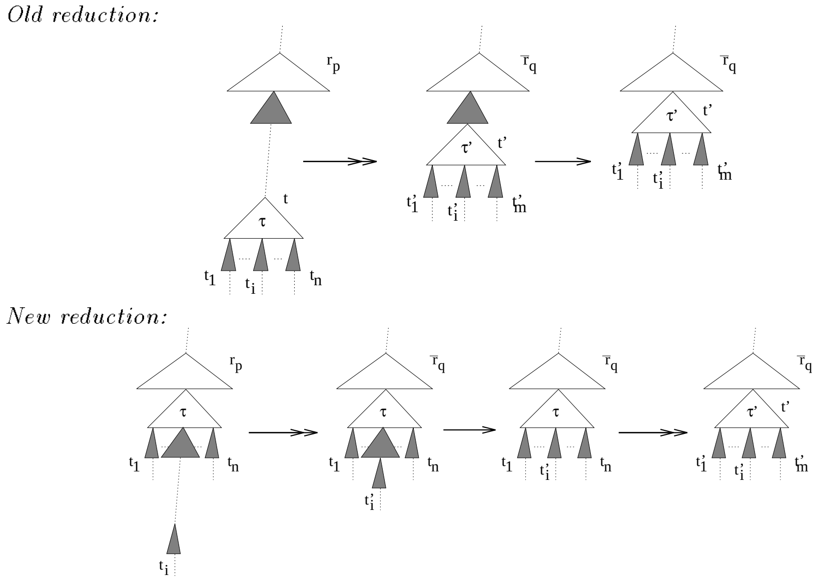


Fig. 3. One case of mimicking.

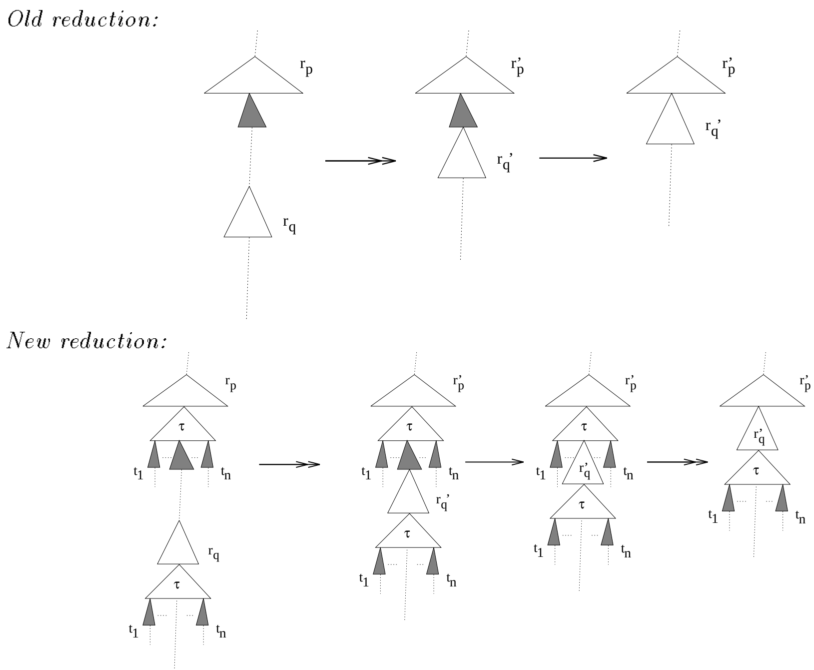


Fig. 4. An application of a deleting sequence.