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On the Modularity of Normal Forms in Rewriting

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The last open problem regarding the modularity of the fundamental properties of Term Rewriting Systems concerns the property of uniqueness of normal forms w.r.t. reduction ($UN^{\rightarrow}$). In this article we solve this open problem, showing that $UN^{\rightarrow}$ is modular for left-linear Term Rewriting Systems. The novel 'pile and delete' technique here introduced allows for quite a short proof, and is of independent interest in the study of modular properties. Moreover, we also study the modularity of consistency w.r.t. reduction ($CON^{\rightarrow}$), showing its modularity for left-linear Term Rewriting Systems.

1. Introduction

Modularity, that is the ability to solve a problem by solving its smaller subparts, is a fundamental topic in modern computer science. Indeed, besides being of interest from a theoretical point of view, modularity has been receiving more and more attention in view of its great potential for practical applications, both for the development and the analysis of large systems.

As far as Term Rewriting Systems (TRSs) are concerned, a property is called modular provided it is valid for two TRSs if and only if it holds for their disjoint union. This area is nowadays a well established theory (see for instance Klop, 1990, 1992; Middeldorp, 1990; Ohlebusch, 1995). It is known of every important property whether it is modular or not, except for one: the last open problem, dating back to 1989 (Middeldorp, 1989; Dershowitz, Jouannaud, and Klop, 1991), regards the modularity of the uniqueness of normal forms with respect to reduction ($UN^{\rightarrow}$ for short).

A TRS is said to have the $UN^{\rightarrow}$ property if every term has at most one normal form. As well known (cf. Middeldorp, 1989), $UN^{\rightarrow}$ is not modular in general: for instance, despite the two TRSs $\{a \rightarrow c, a \rightarrow e, b \rightarrow d, b \rightarrow e, e \rightarrow e\}$ and $\{F(X,X) \rightarrow A\}$ are $UN^{\rightarrow}$, as it is easy to see, in their disjoint union the term $F(a,b)$ has two distinct normal forms, namely $F(c,d)$ and A. However, whether $UN^{\rightarrow}$ is modular when also left-linearity is assumed (that is, when the left hand side of every rewrite rule has all distinct variables) is a question that has been remaining unanswered so far.

As said, in this article we give a solution to this open problem, showing that $UN^{\rightarrow}$ is a modular property for left-linear TRSs.

First, a suitable definition of modular marking of a term is introduced; this naturally

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leads to the formulation of the key concept of modular collapsing (m-collapsing), that will prove to be essential. Indeed, it is shown that, provided only that the TRS is left-linear, failure of $\mathrm{UN}^{\rightarrow}$ cannot occur without m-collapsings.
Second, the strategy we follow is not to analyze the complex behaviour that a general reduction in the disjoint union of two TRSs can have, but instead to modify the reduction in order to get a simpler one: using a novel technique called 'pile and delete', every possible counterexample to the modularity of $\mathrm{UN}^{\rightarrow}$ is translated into one without m-collapsings, thus obtaining a contradiction.
This technique, besides allowing for a rather concise proof, turned out to be important on its own. Since its application does not require the full power of $\mathrm{UN}^{\rightarrow}$ but the weaker property of *consistency with respect to reduction* ($\mathrm{CON}^{\rightarrow}$), stating that a term cannot be rewritten to two different variables, the same proof here given also yields the result that $\mathrm{CON}^{\rightarrow}$ is modular for left-linear Term Rewriting Systems.
Also, thanks to it also a new easy and short proof of the modularity of completeness (see Toyama, Klop, and Barendregt, 1989, 1995) has been given in (Marchiori, 1995a) (even more: see Section 5).
Moreover, the technique has been recently extended in (Marchiori, 1995b) into a general framework, called neatening, which provides a unique, uniform method able to easily prove all the existing results on the modularity of every basic property of left-linear Term Rewriting Systems.

The article is organized as follows: after giving the necessary preliminaries in Section 2, Section 3 introduces the concepts of modular marking and modular collapsing, showing their relevance in the study of $\mathrm{UN}^{\rightarrow}$. Section 4 proves the main theorem stating the modularity of $\mathrm{UN}^{\rightarrow}$ for left-linear TRSs by means of the 'pile and delete' technique. Finally, Section 5 shows that, via the same proof, $\mathrm{CON}^{\rightarrow}$ is modular for left-linear TRSs as well, and examines the modular behaviour of other various weakenings of $\mathrm{UN}^{\rightarrow}$.

2. Preliminaries

The notation used is essentially the one in (Klop, 1992) and (Middeldorp, 1990).

We denote the fixed set of variables as $\mathcal{V}$, and the set of terms built from some signature Σ and $\mathcal{V}$ as $\mathcal{T}(\Sigma, \mathcal{V})$.

The *root* symbol of a term $t \in \mathcal{T}(\Sigma, \mathcal{V})$ is f if $t = f(t_1, \ldots, t_n)$, and t itself otherwise.

When talking about terms, we need also a way to manipulate the subterms contained in them. So, given a signature Σ, a Σ-*context* (*context* for short) is a term in $\mathcal{T}(\Sigma \cup \{\Box\}, \mathcal{V})$, where $\Box$ is a special new symbol (which, intuitively, denotes an 'empty place'). If C is a context with n occurrences of $\Box$, and $t_1, \ldots, t_n$ are terms, then $C[t_1, \ldots, t_n]$ denotes the term obtained from C by replacing from left to right the occurrences of $\Box$ with $t_1, \ldots, t_n$. For instance, if $C = g(\Box, h(a, \Box))$, then $C[a, b] = g(a, h(a, b))$.

A *term rewriting system* (TRS) $\mathcal{R}$ consists of a a signature $\Sigma_{\mathcal{R}}$ and a set of rewrite rules (sometimes called simply rules). A rewrite rule is an object of the form $l \rightarrow r$, where l and r are terms from $\mathcal{T}(\Sigma_{\mathcal{R}}, \mathcal{V})$, such that l is not a variable and all the variables of r appear also in l. l and r are called respectively the left hand side and the right hand side of the rule.

A rewrite rule is is called *left-linear* if in the left hand side every variable does not occur more than once (e.g. $f(g(X, g(Y, Z)) \rightarrow g(X, X))$. It is called *collapsing* if the right hand side is a variable (e.g. $f(X) \rightarrow X$). It is called *duplicating* if there is a variable which occurs more times in the right hand side than in the left hand side (e.g. $f(X) \to g(X,X)$). It is called *erasing* if there is a variable in the left hand side which is not present in the right hand side (e.g. $g(X,Y) \to f(X)$). Also, we say a rule is *non-collapsing* (resp. *non-duplicating*, *non-erasing*) if it is not collapsing (resp. duplicating, erasing). Analogously, a term rewriting system is left-linear, non-collapsing, non-duplicating, non-erasing if each of its rewrite rules is respectively left-linear, non-collapsing, non-duplicating, non-erasing.

A term rewriting system $\mathcal{R}$ determines a rewrite relation $\to_{\mathcal{R}}$ on $\mathcal{T}(\Sigma_{\mathcal{R}}, \mathcal{V})$, defined this way. Given two terms t and t', $t \to_R t'$ if $t = C[l\sigma]$ and $t' = C[r\sigma]$, for some context C, substitution σ, and rewrite rule $l \to r$ in $\mathcal{R}$. If $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \ldots \to_{\mathcal{R}} t_n$ $(n > 0)$, then we say that t_0 *reduces to* t_n in $\mathcal{R}$; correspondingly, we call a *reduction* the sequence $t_0, t_1, \ldots, t_n$, together with the information on what rewrite rule $l_i \to r_i$ has been used to reduce t_i to t_{i+1} $(0 \leq i < n)$, and where it has been applied in t_i (i.e. what subterm of t_i the rule rewrites).

$\twoheadrightarrow_{\mathcal{R}}$ denotes the transitive and reflexive closure of $\to_{\mathcal{R}}$. The *convertibility relation* $\twoheadleftarrow\!\!\twoheadrightarrow_{\mathcal{R}}$ is the transitive, reflexive and symmetric closure of $\to_{\mathcal{R}}$: we will then say that two terms t and t' are convertible if $t \twoheadleftarrow\!\!\twoheadrightarrow_{\mathcal{R}} t'$. When $\mathcal{R}$ is clear from the context, we will simply write Σ, $\to$, $\twoheadrightarrow$ and $\twoheadleftarrow\!\!\twoheadrightarrow$ in place of $\Sigma_{\mathcal{R}}$, $\to_{\mathcal{R}}$, $\twoheadrightarrow_{\mathcal{R}}$, and $\twoheadleftarrow\!\!\twoheadrightarrow_{\mathcal{R}}$.

A term t is in *normal form* for a TRS $\mathcal{R}$ if there is no other term t' such that $t \to t'$ (i.e., t cannot be reduced).

A TRS $\mathcal{R}$ is said to have *unique normal forms w.r.t. reduction* (briefly, to be $\mathrm{UN}^{\to}$), if every term reduces to at most one normal form in $\mathcal{R}$. It is said *consistent w.r.t. reduction* ($\mathrm{CON}^{\to}$) if every term cannot reduce to two different variables.

2.1. MODULARITY

When two term rewriting systems $\mathcal{A}$ and $\mathcal{B}$ have disjoint signatures, we denote with $\mathcal{A} \oplus \mathcal{B}$ their *disjoint union*, that is to say the TRS having as signature the union of the signatures $\Sigma_{\mathcal{A}}$ and $\Sigma_{\mathcal{B}}$, and as rewrite rules both the rewrite rules of $\mathcal{A}$ and those of $\mathcal{B}$. A property $\mathcal{P}$ of term rewriting systems is said *modular* if for every couple of TRSs $\mathcal{A}$ and $\mathcal{B}$ with disjoint signatures, $\mathcal{A} \in \mathcal{P}, \mathcal{B} \in \mathcal{P} \Leftrightarrow \mathcal{A} \oplus \mathcal{B} \in \mathcal{P}$.

Throughout the article we will indicate with $\mathcal{A}$ and $\mathcal{B}$ the two TRSs to operate on. When not otherwise specified, all symbols and notions not having a TRS label are to be intended operating on the disjoint union $\mathcal{A} \oplus \mathcal{B}$. For better readability, we will talk of function symbols belonging to $\mathcal{A}$ and $\mathcal{B}$ like *white* and *black* functions, indicating the first ones with upper case functions, and the second ones with lower case. Variables, instead, have no colour.

Let $t = C[t_1, \ldots, t_n] \in \mathcal{T}(\Sigma_{\mathcal{A}} \cup \Sigma_{\mathcal{B}}, \mathcal{V})$ and $C \neq \Box$; we write $t = C[\![t_1, \ldots, t_n]\!]$ if C is an $\Sigma_{\mathcal{A}}$-context and each of the t_i has $root(t_i) \in \Sigma_{\mathcal{B}}$, or vice versa (exchanging $\mathcal{A}$ and $\mathcal{B}$). The *topmost homogeneous part* (briefly *top*) of a term $C[\![t_1, \ldots, t_n]\!]$ is the context C.

Definition 2.1 The *rank* of a term $t \in \mathcal{T}(\Sigma_{\mathcal{A}} \cup \Sigma_{\mathcal{B}}, \mathcal{V})$ is 1 if $t \in \mathcal{T}(\Sigma_{\mathcal{A}}, \mathcal{V})$ or $t \in \mathcal{T}(\Sigma_{\mathcal{B}}, \mathcal{V})$, and $\max_{i=1}^{n}\{rank(t_i)\} + 1$ if $t = C[\![t_1, \ldots, t_n]\!]$ $(n > 0)$. □

The next well known lemma will be implicitly used in the following:

Lemma 2.2 $s \twoheadrightarrow t \Rightarrow rank(s) \geq rank(t)$

Proof Clear. □

Definition 2.3 The multiset $S(t)$ of the *special subterms* of a term t is

1 $S(t) = \begin{cases} \{t\} & \text{if } t \in (\mathcal{T}(\Sigma_{\mathcal{A}}, \mathcal{V}) \cup \mathcal{T}(\Sigma_{\mathcal{B}}, \mathcal{V})) \backslash \mathcal{V} \\ \emptyset & \text{if } t \in \mathcal{V} \end{cases}$

2 $S(t) = \cup_{i=1}^{n} S(t_i) \cup \{t\}$ if $t = C[\![t_1, \ldots, t_n]\!]$ $(n > 0)$ □

Note that this definition is slightly different from the usual ones in the literature (for example in Middeldorp, 1990), since here variables are not considered to be special subterms.

If $t = C[\![t_1, \ldots, t_n]\!]$, the t_i are called the *principal* special subterms of t. Furthermore, a reduction step of a term t is called *outer* if the rewrite rule is not applied in the principal special subterms of t.

A (strict) *partial order* on the special subterms of a term can be naturally given defining $t_1 \succ t_2$ iff t_2 is a proper special subterm of t_1.

The following proposition will reveal useful:

Proposition 2.4 *If $\mathcal{A}$ and $\mathcal{B}$ are left-linear, then rewrite rules that have the possibility to act outer on a special subterm t are exactly those that have the possibility to act on its top.*

Proof Let $t = C[\![t_1, \ldots, t_n]\!]$: since $t_1, \ldots, t_n$ have a root belonging to the other TRS (with respect to C), they are matched by variables from any rewrite rule applicable to C, and for the left-linearity assumption these variables are independent each other. □

Note that left-linearity is essential for this proposition.

3. Marking and Collapsing

To be able to describe the special subterms of a given term throughout a reduction, it is natural to develop a concept of (modular) marking. A first, naïve approach of modular marking for a term is to take an assignment from the multiset of its special subterms to a (fixed) set of markers. So, for instance, given the term $F(f(G, a), H)$, we could mark $F(\Box, H)$ to m_1, $f(\Box, a)$ to m_2, G to m_3. Then reductions steps, as usual, should preserve the markers.

However, this simple definition presents a problem, since for one case there is ambiguity: when a collapsing rule makes an *inner* top vanish. In this case, we have the situation illustrated in Figure 1, where there is a conflict between m_1 and m_4.

This situation is dealt with by defining a *modular marking* for a term to be an assignment from the multiset of its special subterms to *sets* of markers, and taking in the ambiguous case just described the union of the marker sets of the two special subterms involved.

Thus, the previous example would give what shown in Figure 2 (singletons like $\{m_3\}$ are written simply m_3).

When this situation occurs, we say that the special subterm m_4 has been *absorbed* by m_1, and the special subterm m_2 has had a *modular collapsing* (briefly *m-collapsing*). This last concept will reveal to be crucial in the study of the $\mathrm{UN}^{\rightarrow}$ property (cf. Theorem 3.3).

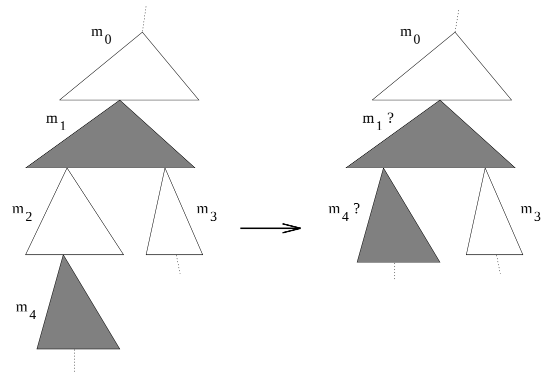

Figure 1. Naïve modular marking.

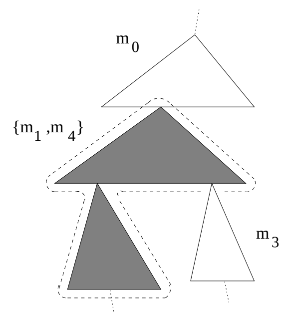

Figure 2. Correct modular marking.

When dealing with reductions $t \twoheadrightarrow t'$ we will always assume, in order to distinguish all the special subterms, that the initial modular marking of t is injective and maps special subterms to singletons.

Inside a reduction a notion of descendant for every special subterm can be defined: in a reduction a special subterm is a *descendant* (resp. *pure descendant*) of another if the set of markers of the former contains (resp. is equal to) the set of markers of the latter. Note, en passant, that due to the presence of duplicating rules, there may be more than one descendant, or even none (due to erasing rules).
Summing up, a special subterm, when a reduction step is applied, can only: i) be erased ii) m-collapse iii) be preserved (i.e. have descendants).
Observe also that, since in a reduction without m-collapsings all the descendants are pure, the first special subterm to m-collapse in a generic reduction is a pure descendant. Hence it readily holds the following:

Fact 3.1 *A reduction has m-collapsings iff a pure descendant m-collapses.*

3.1. LEFT-LINEARITY AND $\mathrm{UN}^{\rightarrow}$

When the left-linearity and $\mathrm{UN}^{\rightarrow}$ properties are introduced, m-collapsings enjoy some remarkable properties. First of all, they behave in a 'deterministic' way, in the following sense:

Proposition 3.1 *Let $\mathcal{A}$ be left-linear and* $\mathrm{UN}^{\rightarrow}$*, and* $t = C[\![t_1, \ldots, t_n]\!]$ *a top white special subterm. Then, if t m-collapses into t_i ($1 \leq i \leq n$) via a white reduction (i.e. using only rules from $\mathcal{A}$), the index i is unique.*

Proof Since $\mathcal{A}$ is left-linear, by Proposition 2.4 the white reduction depends only on the top of t. Hence, if we take instead of $t = C[\![t_1, \ldots, t_n]\!]$ a term $t' = C[\![X_1, \ldots, X_n]\!]$ (with $X_1, \ldots, X_n$ new fresh variables), then every previous white reduction that m-collapsed t to t_i can be repeated on t' to reduce it to X_i, and if the index i were not unique t' could be reduced to different normal forms, contradicting the fact $\mathcal{A}$ is $\mathrm{UN}^{\rightarrow}$. □

Moreover, the concept of m-collapsing reveals to be crucial in the study of $\mathrm{UN}^{\rightarrow}$ modularity for the following reason:

Definition 3.2 A $\mathrm{UN}^{\rightarrow}$ *counterexample* (briefly *counterexample*) is a pair (d_1, d_2), where $d_1 : s \twoheadrightarrow n_1$ and $d_2 : s \twoheadrightarrow n_2$ are reductions starting from the same term s (called the *start*) and ending in two normal forms $n_1 \neq n_2$ (called the *ends*). □

Theorem 3.3 *If $\mathcal{A}$ and $\mathcal{B}$ are left-linear and* $\mathrm{UN}^{\rightarrow}$*, then there is no counterexample without m-collapsings.*

Proof Take a reduction without m-collapsings ending in a normal form. Every rule acts on the top of a well specified special subterm, and this top cannot change since no m-collapsing is present. Moreover, by Proposition 2.4, the application of these rules depends only on the top itself. So for every top of a special subterm a separate reduction is performed, that must eventually lead in the ends to a unique top for the $\mathrm{UN}^{\rightarrow}$ property, and hence the resulting normal form is unique as well. □

4. Pile and Delete

The pile and delete technique here employed allows, once given a term and some reductions that normalize it, to transform the given term (and correspondingly the reductions too) in such a way to preserve the set of normal forms previously obtained, but this time with reductions in a nice form, that is without m-collapsings.

Proposition 4.1 *If $\mathcal{A}$ and $\mathcal{B}$ are left-linear and* $\mathrm{UN}^{\rightarrow}$*, every counterexample can be translated into a counterexample without m-collapsings.*

Proof If the counterexample is already without m-collapsings, the assertion is trivially satisfied. So, suppose it is not. Select a special subterm of the start of the counterexample that has rank minimal amongst the ones with a pure descendant that m-collapses in the counterexample itself: say $t = \tau[\![t_1, \ldots, t_n]\!]$.
This special subterm cannot have a pure descendant in the ends of the counterexample. Indeed, suppose it is so, and t reaches a normal form n. Because of its rank minimality, t must m-collapse by Proposition 3.1 into a fixed principal subterm, namely t_i. So, substituting (in t) t_i with a new fresh variable X, we can obtain by Proposition 2.4 a reduction from this new term t' to the normal form X which is *without* m-collapsings. On the other hand, t also reduces to the normal form n via a reduction without m-collapsings (again, by the minimality assumption) and so, by Proposition 2.4, disregarding what is in t_i: therefore, also t' reduces to n via the same reduction. So, t' reduces both to X and to n (which is different from X), hence giving a counterexample without m-collapsings, in contradiction with Theorem 3.3.

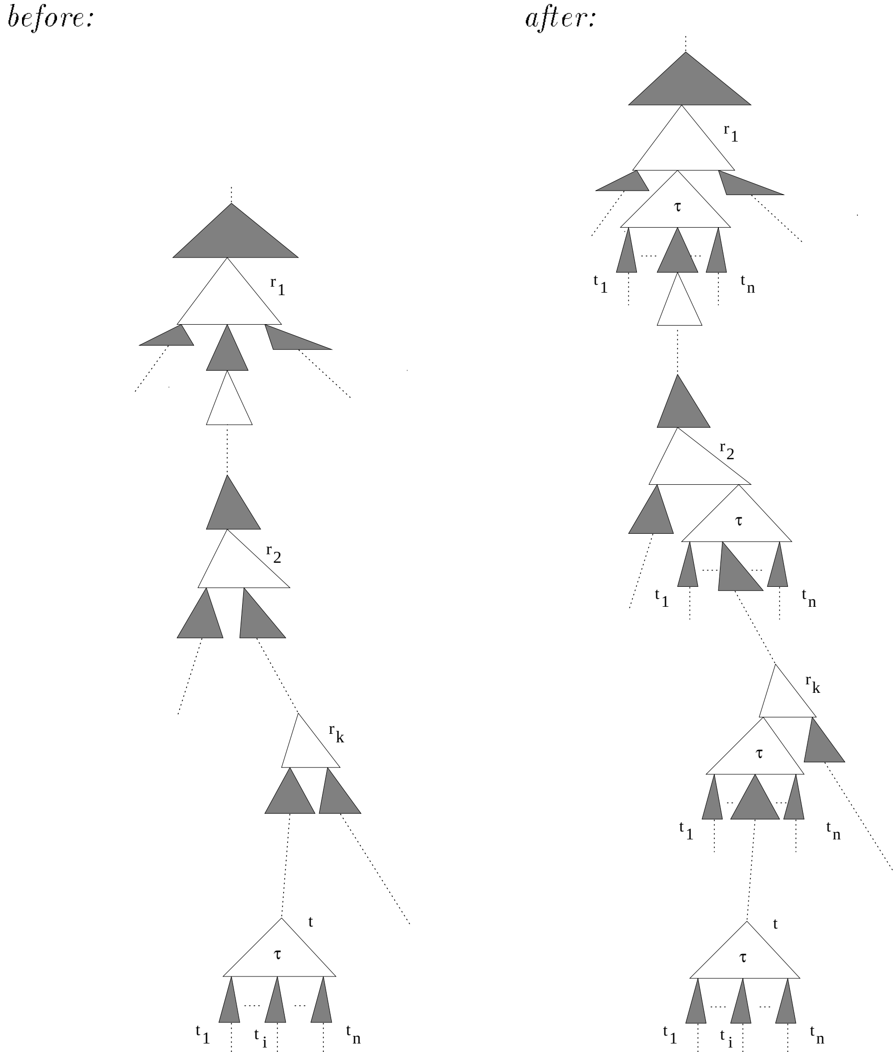

Figure 3. The 'pile' process.

The fact that t alone cannot reach the ends does not mean that its top, τ, is not needed at all in the counterexample: it may be needed, via absorption, from other white tops of $(\prec)$-greater special subterms in the counterexample. All of these special subterms $\bar{r}_1, \ldots, \bar{r}_\ell$ are descendants of some special subterms of the start $r_1, \ldots, r_k$ $(k \leq \ell)$.
We can so try to perform 'in advance' these absorptions, modifying directly the start of the counterexample, using the following 'pile and delete' technique.
First, we 'pile' $\tau[\![t_1, \ldots, t_{i-1}, \Box, t_{i+1}, \ldots, t_n]\!]$ just below the tops of the $r_1, \ldots, r_k$. That is to say if $r_i = r_i[\![s_1, \ldots, s_v]\!]$ and t is in s_j (viz. $t \prec s_j$), then r_i is replaced with

$$r_i[\![s_1, \ldots, s_{j-1}, \tau[t_1, \ldots, t_{i-1}, s_j, t_{i+1}, \ldots, t_n], s_{j+1} \ldots, s_v]\!]$$

The situation is illustrated in Figure 3.

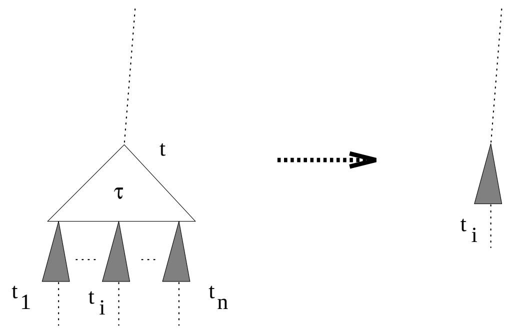

Figure 4. The 'delete' process.

Intuitively, the top of t is not really needed any more, since we have already inserted copies of it where needed for absorption, and it has been proved earlier that t alone cannot stay till an end of the counterexample: therefore we 'delete' it replacing t by t_i (see Figure 4).
Now it has to be shown that the original counterexample can still be mimicked using this revised start term; this can be done because we can get rid of the piled τ, when not needed, using the original reduction from the counterexample that m-collapsed it ($t \twoheadrightarrow t_i$):

- By minimality of t, the only effect of the rules acting on the pure descendants of t but not on the pure descendants of t_i was to m-collapse t into a descendant of t_i (if this is not the case, then it means that the descendant of t must be erased), and so they can be dropped since we already replaced t with t_i.
- When a descendant of t was absorbed by, say, $\bar{r}_q$, we have piled to its ancestor r_p (and so to its descendant $\bar{r}_q$) in that place $\tau[\![t_1, \ldots, t_{i-1}, \Box, t_{i+1}, \ldots, t_n]\!]$, whereas the old descendant of t is now the corresponding descendant of t_i. So it only remains to reduce the piled $\tau[\![t_1, \ldots, t_{i-1}, \Box, t_{i+1}, \ldots, t_n]\!]$ as previously in the counterexample to obtain exactly the same situation as before, and the new counterexample can proceed in the mimicking (see Figure 5).
  Note how these postponed reductions produce no m-collapsings.
- We inserted $\tau[\![t_1, \ldots, t_{i-1}, \Box, t_{i+1}, \ldots, t_n]\!]$ below all the $r_1, \ldots, r_k$, but actually pure descendants of t may be absorbed in the initial counterexample only by part of the descendants of these special subterms. However, we can get rid of these superfluous occurrences of material acting, as hinted previously, with the rules that in the initial counterexample made $\tau[\![t_1, \ldots, t_{i-1}, \Box, t_{i+1}, \ldots, t_n]\!]$ collapse into $\Box$: they are applied to all of these extra descendants when the piled material is not needed any more. This means that these 'deleting sequences' must be applied
  i) when in the sequel of the original reduction the descendant of an r_p will not absorb a pure descendant of t any more, or
  ii) when the descendant of an r_p absorbs another descendant of an r_q (Figure 6).

  Again, it is immediate to see that these deleting sequences produce no m-collapsings.

This way we have obtained a new counterexample with a different start but the same ends as the initial one. Once again, note that left-linearity, via Proposition 2.4, was essential to be able to mimic the old counterexample.

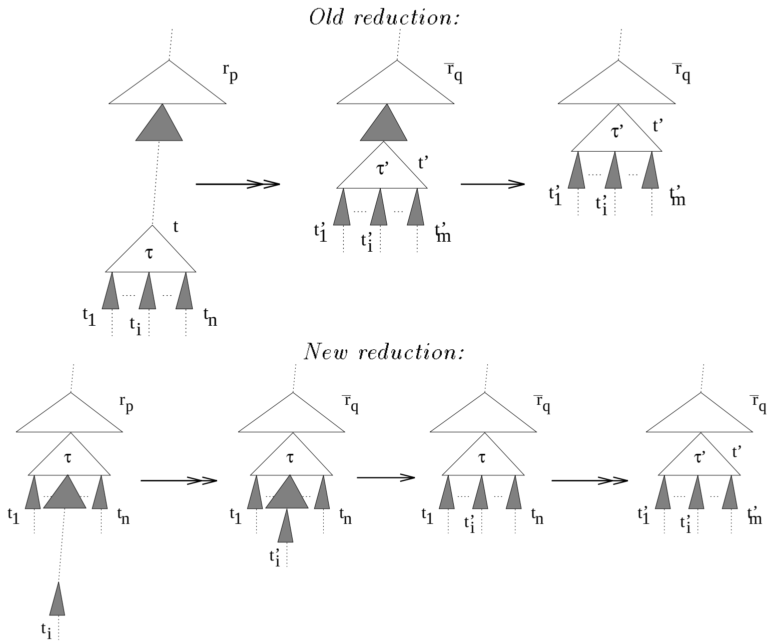

Figure 5. One case of mimicking.

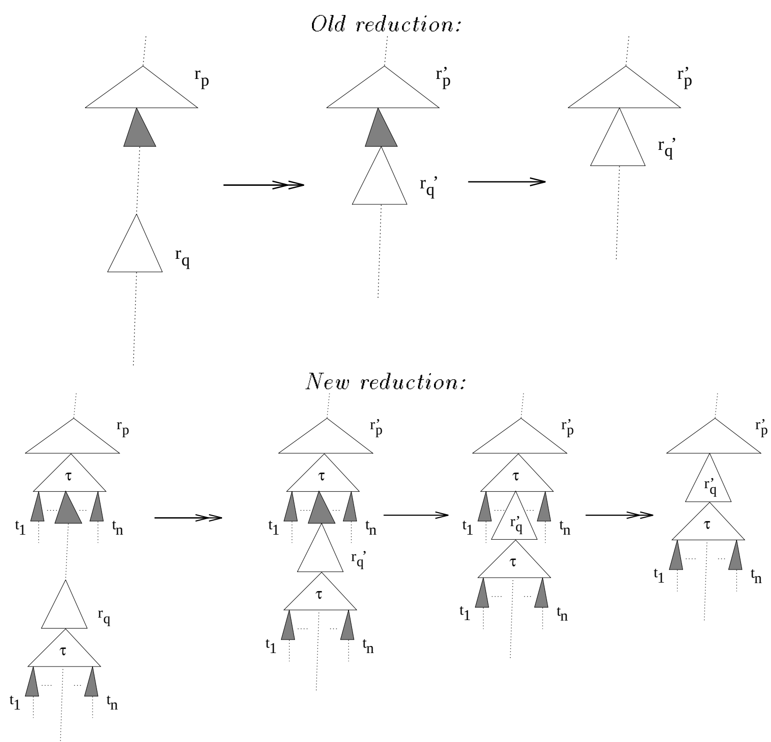

Figure 6. An application of a deleting sequence.

Now consider the number of special subterms of the start with a descendant that m-collapses in the counterexample itself: this new counterexample obtained via the pile and delete technique has this number diminished at least by one with respect to the initial counterexample; indeed, t is no more present, and as remarked no new m-collapsings are introduced modifying the original reductions.

So, repeating this 'pile and delete' process leads, ultimately, to a counterexample without m-collapsings. □

Example 4.2 Consider the two following (left-linear and UN$^{\rightarrow}$) TRSs

$$\mathcal{A} = \begin{cases} F(X) \to G(X,X) \\ G(L(X,Y),Z) \to Y \\ H(X,Y) \to L(X,Y) \\ A \to B \end{cases} \qquad \mathcal{B} = \begin{cases} f(X) \to X \\ g(f(X)) \to a \\ g(X) \to g(X) \end{cases}$$

and the reduction (unary functions like $f(A)$ are for short written fA from now on)

$$gFfH(A,fA) \to gFfH(A,fB) \to gG(fH(A,fB),fH(A,fB)) \to$$
$$gG(fH(A,fB),H(A,fB)) \to gG(fH(A,fB),H(A,B)) \to$$
$$gG(H(A,fB),H(A,B)) \to gG(L(A,fB),H(A,B)) \to gfB \to a$$

The special subterm of the starting term with minimal rank among the ones that m-collapse in this reduction is fA. Thus, after the pile and delete process we get

$$gfFffH(A,A) \to gfFffH(A,B) \to gfG(ffH(A,B),ffH(A,B)) \to$$
$$gfG(ffH(A,B),fH(A,B)) \to gfG(ffH(A,B),H(A,B)) \to$$
$$gfG(fH(A,B),H(A,B)) \to gfG(H(A,B),H(A,B)) \to$$
$$gfG(L(A,B),H(A,B)) \to gfB \to a$$

Now the minimal special subterm of the starting term that m-collapses is $ffH(A,A)$; the corresponding reduction after the pile and delete is

$$gfffFH(A,A) \to gffFH(A,A) \to gfFH(A,A) \to gfFH(A,B) \to$$
$$gfG(H(A,B),H(A,B)) \to gfG(L(A,B),H(A,B)) \to gfB \to a$$

and this reduction is without m-collapsings. □

Theorem 4.3 UN$^{\rightarrow}$ *is a modular property for left-linear TRSs.*

Proof One implication is obvious. On the other hand, if $\mathcal{A}$ and $\mathcal{B}$ are UN$^{\rightarrow}$ but $\mathcal{A}\oplus\mathcal{B}$ is not, then it has a counterexample that can be translated into a counterexample without m-collapsings by the above Proposition 4.1, contradicting Theorem 3.3. □

5. Weakening UN$^{\rightarrow}$

The 'pile and delete' technique does not need the full power of UN$^{\rightarrow}$, but it can be applied under the weaker assumption of *consistency with respect to reduction* (briefly CON$^{\rightarrow}$), that is satisfied if every term cannot be rewritten to two different variables.

This is true since the 'pile and delete' technique essentially relies upon Proposition 3.1, that still holds if $\mathrm{CON}^{\rightarrow}$ is required in place of $\mathrm{UN}^{\rightarrow}$. Hence, if we replace the definition of $\mathrm{UN}^{\rightarrow}$-counterexample with the corresponding definition of $\mathrm{CON}^{\rightarrow}$-counterexample (where the ends are required to be variables), exactly the same proof here used for the modularity of $\mathrm{UN}^{\rightarrow}$ shows that

Theorem 5.1 $\mathrm{CON}^{\rightarrow}$ *is a modular property for left-linear TRSs.*

This result, together with the modularity of $\mathrm{UN}^{\rightarrow}$ for left-linear TRSs, completes an interesting parallelism between the pairs (UN, $\mathrm{UN}^{\rightarrow}$) and (CON, $\mathrm{CON}^{\rightarrow}$) (a TRS is *consistent*, CON for short, if different variables cannot be convertible; analogously, a TRS has the *unique normal form* property (UN) if different normal forms are not convertible). Indeed, this parallelism is present in all the other cases, since:

(i) $\mathrm{UN} \Rightarrow \mathrm{UN}^{\rightarrow}$ and $\mathrm{UN} \not\Leftarrow \mathrm{UN}^{\rightarrow}$:
the first implication is straightforward, while for the second fact (cf. Middeldorp, 1989), the TRS $\{a \to b, a \to c, c \to c, d \to c, d \to e\}$ is $\mathrm{UN}^{\rightarrow}$ but not UN

(ii) $\mathrm{CON} \Rightarrow \mathrm{CON}^{\rightarrow}$ and $\mathrm{CON} \not\Leftarrow \mathrm{CON}^{\rightarrow}$:
again, the first implication is trivial; for the second fact, take the TRS $\{f(X) \to X, f(X) \to a\}$ that is $\mathrm{CON}^{\rightarrow}$ but $X \leftarrow f(X) \to a \leftarrow f(Y) \to Y$

(iii) UN is modular unlike $\mathrm{UN}^{\rightarrow}$:
the modularity of UN has been proved in (Middeldorp, 1989), and a counterexample to the modularity of $\mathrm{UN}^{\rightarrow}$ can be found in the introduction of this paper

(iv) CON is modular unlike $\mathrm{CON}^{\rightarrow}$:
the modularity of CON has been proved in (Schmidt-Schauß, 1989), whereas to see that $\mathrm{CON}^{\rightarrow}$ is not modular take the two TRSs $\{f(X) \to X, f(X) \to a\}$ and $\{F(X,X,Y) \to Y, F(X,Y,Y) \to X\}$ that are $\mathrm{CON}^{\rightarrow}$ but in their disjoint union $X \leftarrow f(X) \leftarrow F(f(X),a,a) \leftarrow F(f(X),a,f(Y)) \to F(a,a,f(Y)) \to f(Y) \to Y$

Furthermore, using the fact that $\mathrm{CON}^{\rightarrow}$ suffices to apply the pile and delete technique, exactly the same pile and delete technique here employed has been utilized in (Marchiori, 1995a) not only to give a new easy and short proof of the deep result in (Toyama, Klop, and Barendregt, 1989, 1995) stating the modularity of completeness for left-linear TRSs, but also to *extend* the result there proved showing the modularity of termination for left-linear and consistent with respect to reduction TRSs.
In this connection, Theorem 5.1 is extremely useful since it allows to lift that result to an arbitrary number of TRSs: if T_1 and T_2 are left-linear, $\mathrm{CON}^{\rightarrow}$ and terminating then $T_1 \oplus T_2$ is again left-linear (obvious), $\mathrm{CON}^{\rightarrow}$ (by Theorem 5.1) and terminating (by the aforementioned result), hence we can repeat this reasonment to prove termination for the disjoint union of an arbitrary number of TRSs $T_1, \ldots, T_n$.

Other weakenings of $\mathrm{UN}^{\rightarrow}$ do not show to have a good modular behaviour. Indeed, consider the property k-$\mathrm{UN}^{\rightarrow}$ ($k \geq 1$), satisfied if every term has at most k normal forms. Then for $k = 1$ we get just $\mathrm{UN}^{\rightarrow}$, and for $1 \leq i < j$ we have i-$\mathrm{UN}^{\rightarrow} \Rightarrow j$-$\mathrm{UN}^{\rightarrow}$ but not vice versa (as it is trivial to check). All the weaker properties k-$\mathrm{UN}^{\rightarrow}$ ($k > 1$) are not modular even for left-linear TRSs, as shown by the following counterexample. Let $T_1^{(k)} = \{a \to a_1, \ldots, a \to a_k\}$ and $T_2^{(k)} = \{F(X,Y) \to G(X,Y)\}$. Then

$T_1^{(k)}$ and $T_2^{(k)}$ are left-linear, k-UN$^{\rightarrow}$ and even non-erasing and non-collapsing. Nevertheless, $T_1^{(k)} \oplus T_2^{(k)}$ is not k-UN$^{\rightarrow}$, since $F(a,a)$ has k^2 normal forms corresponding to $G(a_1,a_1), G(a_1,a_2), \ldots, G(a_k,a_k)$. Note one cannot even provide a bound on the number of normal forms, since $F(a, F(a, \cdots, F(a,a)\cdots))$ (with n occurrences of F) has k^{n+1} normal forms.

Finally, let us conclude saying that the main results here proved in Theorems 4.3 and 5.1 do not hold for the more general combinations of TRSs so far studied, where the signatures can somehow overlap (see e.g. Kurihara and Ohuchi, 1991; Klop, 1992; Ohlebusch, 1995). Even in the limited case of constructor-sharing systems (TRSs that can share 'constructors', i.e. symbols not present at the top of the left hand side in some rewrite rule) there is a counterexample: $T_1 = \{F(C(X),Y,Z) \rightarrow Y, F(D(X),Y,Z) \rightarrow Z\}$ and $T_2 = \{a \rightarrow C(a), a \rightarrow D(a)\}$ are left-linear and UN$^{\rightarrow}$, and share only the constructors C and D, but their union is not CON$^{\rightarrow}$ (and so, a fortiori, also not UN$^{\rightarrow}$), since we have the reductions $Y \leftarrow F(C(a),Y,Z) \leftarrow F(a,Y,Z) \rightarrow F(D(a),Y,Z) \rightarrow Z$.

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