The Theory of Vaccines

Massimo Marchiori
Department of Pure and Applied Mathematics
University of Padova, Via Belzoni 7, 35131 Padova, Italy
Phone: +39 49 8275972 Fax: +39 49 8758596
max@math.unipd.it

Abstract

Despite the major role that modularity occupies in computer science, all the known results on modular analysis only treat particular problems, and there is no general unifying theory. In this paper we provide such a general theory of modularity. First, we study the space of the criteria for modularity (the so-called modularity space), and give results on its complexity. Then, we introduce the notion of vaccine and show how it can be used to completely analyze the modular space. It is also shown how vaccines can be effectively used to solve a variety of other modularity problems, providing the best solutions. As an application, we successfully apply the theory to the study of modularity for term rewriting, giving for the first time optimality results, and completely solving the modularity problem for the major properties of rewriting.

AMS Classification: 68N05, 68Q42, 68Q60
CR Categories: D.2.4, D.2.7, D.3.3, D.4.7, F.3.1, F.3.3, F.4.2.
Keywords: term rewriting, modularity, optimality, program verification, program development.

1 Introduction

The field of modular analysis is of fundamental importance, and is nowadays attracting increasing interest by the scientific community. In essence, modularity allows to study a complex object by studying his smaller subparts: given a 'big' object composed by smaller subparts (via some composition operator), we want to state that it enjoys a certain property by simply investigating its smaller subcomponents. Hence, modular analysis allows to develop correct complex objects 'bottom-up', just building correct smaller submodules, and even dually to verify the correctness of a complex object by decomposing it into its submodules and verifying them.

Besides for the theoretical relevance, the increasing complexity of nowadays applications has made modularity analysis a task of primary importance from the practical side as well.

At the present moment, the field of modular analysis consists of several results that study the modularity of a particular property for a certain specific paradigm (see e.g. [12, 4, 27, 1, 19, 13, 8, 22, 10, 24]). However, there is no general theory on modular analysis. In this paper, we introduce such a theory.

Given the property to be verified, and the 'composition operator' that builds complex objects from smaller submodules, we analyze the corresponding *modularity space*, that is to say the collection of all the criteria for the modularity of the property w.r.t. the composition operator.

First, a complete description of this space by means of its maximal criteria is provided (roughly speaking, the 'best' results that can be obtained), and its complexity is studied (how many maximal criteria can exist). Next, we introduce the notion of *vaccine*, which is used for analyzing in an effective way the modularity space. Intuitively, a vaccine extracts from a possibly non-modular property a maximal modular sub-property, that is a maximal criterion of the modularity space for that property. Therefore, vaccines provide a convenient way to represent the modularity space. We

propose a methodology for finding vaccines (and so the optimal modularity criteria). Moreover, we provide suitable conditions that ensure that the analysis of the modularity space is *completely solved*, i.e., it covers all the optimal criteria, and consequently every possible modularity criterion (being all the others subsumed by the maximal criteria).

Furthermore, it is shown that an analysis which is completely solved, is relevant for the study of the class of the *disjunctive criteria* (cf. [19, 27]), because it provides the *best* disjunctive criterion.

Finally, we consider also the other side of the coin, namely the case when modularity does not hold. We introduce the notion of *counterexample structure*, which is used together with the notion of vaccine for recovering the *best* description of the failure of modularity. The above results are successfully applied to the study of the modularity of important properties of term rewriting systems: termination, completeness and uniqueness of normal forms (the only main properties of TRSs that are not modular). In particular, we show that $\mathcal{C}_\mathcal{E}$-termination (cf. [10, 21]) is a maximal criterion, and provide a formal justification in terms of complexity of the difficulty of the study of the modularity of termination in TRS. Moreover, we completely solve the problem of the modularity of termination for left-linear TRSs, providing the only two optimal criteria. We give analogous results for the other major properties of completeness and uniqueness of normal forms, thus not only improving on all the works on the modularity of these properties, but completely solving the problem of their modular analysis.

The paper is organized as follows. Section 2 starts with some short preliminaries. Soon afterwards, Section 3 presents the notion of modular analysis and of a criterion for modularity. Then, Section 4 introduces the *modularity space* and gives some results on its complexity. In Section 5 the concept of *vaccine* is introduced. Next, Section 6 shows how vaccines can be successfully employed for the study of the modularity space via the notion of *vaccines basis*. Section 7 analyzes another kind of criteria, the so-called disjunctive criteria, and shows how they can be successfully analyzed via vaccines. Section 8 performs the same task for the study of counterexample structures, giving a complete analysis of the failure of modularity. Sections 9 and 10 present successfully practical applications of the theory for the field of term rewriting. Finally, Section 11 ends with some conclusive remarks.

2 Preliminaries

$\mathcal{O}$ denotes the class of generic objects we will consider: every object is understood to be in $\mathcal{O}$.

As usual, properties of objects will be identified with the classes of objects that belong to them. So, we will write equivalently $\mathcal{Q}_1 \wedge \mathcal{Q}_2$ or $\mathcal{Q}_1 \cap \mathcal{Q}_2$ to denote the intersection of two properties $\mathcal{Q}_1$ and $\mathcal{Q}_2$. We will also write $\neg\mathcal{Q}$ to indicate the complement property of $\mathcal{Q}$ (i.e. $T \in \neg\mathcal{Q}$ iff $T \notin \mathcal{Q}$).

As far as TRSs are concerned, we only require knowledge of the basic notions (see e.g. [5, 12]). The reader interested in modularity topics of TRSs can find extensive surveys in [20, 22].

3 Modularity

Suppose we want to perform the modular (w.r.t. some composition operator $\odot$) analysis of the property $\mathcal{P}$: given a complex object $T_1 \odot \cdots \odot T_n$ we want to infer it belongs to $\mathcal{P}$ by separately analyzing its smaller submodules $T_1, \ldots, T_n$.

The best case occurs when the property $\mathcal{P}$ is *modular* (w.r.t. a binary composition operator $\odot$): whenever n objects $T_1, \ldots, T_n$ are in $\mathcal{P}$, their composition $T_1 \odot \ldots \odot T_n$ is in $\mathcal{P}$ as well. Thus, to check a complex object $T_1 \odot \ldots T_n$ belongs to $\mathcal{P}$, it just suffices to check its submodules $T_1, \ldots, T_n$ belong to $\mathcal{P}$. In general, however, $\mathcal{P}$ may not be modular, and so we need a more general concept to formalize modular analysis. We so define what is the notion of a criterion for modularity:

Definition 3.1 $\mathcal{Q}$ is a *criterion (for the $\odot$-modularity of $\mathcal{P}$)* if $\mathcal{Q} \neq \emptyset$ and $\forall T_1, \ldots, T_n.\ T_1 \in \mathcal{Q}, \ldots, T_n \in \mathcal{Q} \Rightarrow T_1 \odot \ldots \odot T_n \in \mathcal{P}$. □

In the sequel we will often talk simply of criterion, omitting $\mathcal{P}$ and $\odot$.

So, having a criterion $\mathcal{Q}$ we can perform modular analysis of a complex object $T_1 \odot \ldots \odot T_n$ just by separately checking that every submodule belongs to $\mathcal{Q}$.

3.1 Assumptions

Given the property $\mathcal{P}(\neq \emptyset)$ whose modular behaviour we want to analyze, we call *healthy* the objects in $\mathcal{P}$, and *sick* the others (the reasons for this terminology will become clear when we will introduce vaccines in Section 5).

We say that two objects A and B are *compatible* (resp. *uncompatible*) w.r.t. $\mathcal{P}$ and $\odot$, if $A \odot B$ is healthy (resp. sick).

Recall that a *groupoid* $(\mathcal{S}, \beta)$ is a set $\mathcal{S}$ equipped with a binary operation β. Although this is not strictly needed for the development of our theory, for simplicity we suppose that every groupoid we talk about has a neutral element (if it is not the case, one can always be added by the standard lifting technique).

The observable of interest is the modular behaviour of objects w.r.t. the property $\mathcal{P}$. So, we introduce the following relation:

Definition 3.2 Two objects B and C of the groupoid $(\mathcal{O}, \odot)$ are said to be *modularly congruent w.r.t.* $\mathcal{P}$ (notation $B =_{\mathcal{P}} C$), if $\forall A.\ A \odot B \in \mathcal{P} \Leftrightarrow A \odot C \in \mathcal{P}$. □

The intuition is that $B =_{\mathcal{P}} C$ if B and C have the same modular behaviour with respect to $\mathcal{P}$: substituting in a complex object B with C preserves the property $\mathcal{P}$. It can be proved that $=_{\mathcal{P}}$ *is a congruence of* $(\mathcal{O}, \odot)$, hence it makes sense to reason about the *factor groupoid of* $(\mathcal{O}, \odot)$ *w.r.t.* $=_{\mathcal{P}}$ (that is to say, the groupoid obtained by $(\mathcal{O}, \odot)$ by considering equivalence classes modulo $=_{\mathcal{P}}$): $(\mathcal{O}/_{=_{\mathcal{P}}}, \odot/_{=_{\mathcal{P}}})$.

Another crucial definition is the following:

Definition 3.3 A groupoid $(\mathcal{O}, \odot)$ is said to $\mathcal{P}$-*dense* if $\forall T_1, T_2.\ T_1 \odot T_2 \in \mathcal{P} \Rightarrow T_1 \in \mathcal{P} \wedge T_2 \in \mathcal{P}$. □

Roughly speaking, density corresponds to the very reasonable assumption that objects constituting a healthy object are themselves healthy.

Now we have all the ingredients to define this main notion:

Definition 3.4 A $\mathcal{P}$-*acid groupoid* (briefly, a $\mathcal{P}$-*acid*, is a groupoid $(\mathcal{O}, \odot)$ such that $(\mathcal{O}/_{=_{\mathcal{P}}}, \odot/_{=_{\mathcal{P}}})$ is a $\mathcal{P}$-dense semilattice. □

The name "acid" stems from the fact a semilattice can equivalently be seen as an *aci*-groupoid (viz. a groupoid $(\mathcal{S}, \beta)$ that is *a*ssociative, *c*ommutative and *i*dempotent), and so *acid* stands for *aci* and *d*ense.

Note that if $(\mathcal{O}, \odot)$ is an aci-groupoid, then $(\mathcal{O}/_{=_{\mathcal{P}}}, \odot/_{=_{\mathcal{P}}})$ is an aci-groupoid as well, for every property $\mathcal{P}$.

Assumption: Thorough the paper, we assume that $(\mathcal{O}, \odot)$ is a $\mathcal{P}$-acid.

We remark that for most of the results all of the above assumptions are not necessary. We take all of them at once to simplify readability: the results stated with their 'bare-bones' assumptions can be found in the full paper [15].

4 The Modularity Space

The study of modularity for a given healthiness property is tantamount to the study of the criteria for its modularity. We are so interested in the *modular space* (*m-space*), that is in the collection of all the criteria for modularity. A way to express this information is to consider only the most significant objects in this space. The m-space has a natural partial ordering, namely the set inclusion; the idea is so to consider only the tops of the m-space:

Definition 4.1 The *modular basis* (*m-basis* for short) is the collection of all the maximal criteria. The *modular dimension* (*m-dimension*) is the cardinality of the m-basis. □

The modular basis is a good representative of the modular space, since from it we can build up the whole modular space (the maximal criteria entail all the other criteria):

Theorem 4.2 *Every criterion is contained in a maximal criterion.*

4.1 k-counterexamples

The m-dimension gives an abstract measure of the complexity of the modular space. It is not difficult to see that the m-dimension is one iff $\mathcal{P}$ is modular, and if $\mathcal{P}$ is not modular the m-dimension is at least two. We now give more precise results on the m-dimension, introducing the concept of k-counterexample.

Given an ordinal k, a *k-counterexample* (to the $\odot$-modularity of $\mathcal{P}$) is a collection $A_1, \ldots, A_k$ of pairwise uncompatible healthy objects.

Usually, a 2-counterexample will be simply called a *counterexample*.

The next two lemmata provide the link between k-counterexamples and the m-dimension. The first result gives a lower bound:

Lemma 4.3 *If there is a k-counterexample, then the m-dimension is at least k.*

The second result, dually, gives an upper bound:

Lemma 4.4 *If there is not a k-counterexample ($k < \omega$), then the m-dimension is less than k.*

Combining the above bounds gives the following characterization of the m-dimension in the finite case:

Corollary 4.5 *The m-dimension is k ($k < \omega$) iff there is a k-counterexample but there is no $k+1$-counterexample.*

5 Vaccines

We said the basic notion of the theory is that of vaccine. A vaccine is "a preparation of living attenuated organisms, or living fully virulent organisms that is administered to produce or artificially increase immunity to a particular disease" (Webster's 7th Collegiate Dictionary). So, suppose we want to ensure an organism enjoys a particular property. We can inject a specific vaccine for this property to it: if it does not get sick, due to collateral effects, we are sure it is immunized and enjoys that property.

In this paper, we utilize the notion of vaccine in a formal setting to study modularity. Therefore, suppose we want to study the modularity behaviour of the class of objects $\mathcal{P}$. The idea is to consider $\mathcal{P}$ as a 'healthiness condition', and select some representative objects that make things go wrong (i.e. that cause modularity to fail), using them as a vaccines: we can 'inject' one of them, say A, to any other object in $\mathcal{P}$ via the composition operator $\odot$: in case there are no collateral effects, i.e. in case the object is still healthy (belonging to $\mathcal{P}$), it will become 'immunized' to that particular disease that made modularity fail.

More formally, an object A is a vaccine if for the class of its vaccinated objects ($\{T : T \odot A \in \mathcal{P}\}$), $\mathcal{P}$ becomes $\odot$-modular.

The nice fact, as said in the introduction, is that we will show that the criteria defined by vaccines are *optimal* (i.e. maximal). This way, vaccines provide a tool to completely describe the modular space, providing the best criteria.

We now start giving rigorous formal definitions.

Definition 5.1 The class of *objects vaccinated via A with injection operator $\odot$ and healthiness property $\mathcal{P}$* is

$$\mathbf{V}_A^{\odot}(\mathcal{P}) \;=\; \{T\colon T\odot A \in \mathcal{P}\}$$ □

That is, we take every object T and inject A to it, obtaining the healthy object $T\odot A$.

The operator $\odot$ and the healthiness property $\mathcal{P}$ will be mostly omitted and considered understood, hence we will also write simply $\mathbf{V}_A$.

Now, we can define what a vaccine for modularity is:

Definition 5.2 *A is a vaccine (for the $\odot$-modularity of $\mathcal{P}$)* if $\mathbf{V}_A$ is a criterion for the $\odot$-modularity of $\mathcal{P}$. □

That is to say,

$$\emptyset \neq \mathbf{V}_A,\ T_1 \in \mathbf{V}_A, \ldots, T_k \in \mathbf{V}_A \Rightarrow T_1 \odot \ldots \odot T_k \in \mathcal{P}$$

Vaccines can be composed to get new vaccines, as the following results show:

Lemma 5.3 (Composition) *Suppose A is a vaccine for $\mathcal{P}_1$ and B is a vaccine for $\mathcal{P}_2$. If $A\odot B \in \mathcal{P}_1 \wedge \mathcal{P}_2$, then $A\odot B$ is a vaccine for $\mathcal{P}_1 \wedge \mathcal{P}_2$.*

Corollary 5.4 *If A and B are compatible vaccines, then $A\odot B$ is a vaccine.*

Vaccines are only representatives of the corresponding criteria. It is therefore important to ask when different vaccines are representative of the same class. The following lemma gives a neat answer to this question:

Lemma 5.5 *Let A and B be vaccines. Then, $\mathbf{V}_A = \mathbf{V}_B \Leftrightarrow$ A and B are compatible*

6 Vaccines Bases

Every vaccine for modularity defines a criterion for modularity given by the class $\mathbf{V}_A$. The most important reason that makes vaccines attractive to study is that this criterion is *optimal* in the sense that *cannot be improved*.

Theorem 6.1 (Optimality) *If A is a vaccine, then $\mathbf{V}_A$ is a maximal criterion.*

The m-basis is an abstract concept. Anyway, we have just seen that vaccines can conveniently represent the maximal criteria. So, we introduce a new manageable representative of the m-space:

Definition 6.2 A *vaccines basis* (*v-basis*) is a collection of vaccines $\{A_i\}_{i=1\ldots k}$ (k an ordinal) such that every maximal criterion is represented by exactly one vaccine. □

Hence, $A_1, \ldots, A_k$ is a v-basis iff $\mathbf{V}_{A_1}, \ldots, \mathbf{V}_{A_k}$ is the m-basis.

A v-basis does not only give a complete description of the modular space. It also allows to easily derive that a property is indeed a criterion by proving that it is weaker than an optimal criterion. The precise technique is described in the full paper. This also holds for the other kind of criteria, namely d-criteria (cf. Section 7). Hence not only easy proofs of the previously existing results on modularity can be given, but also investigation of new practical criteria is possible.

6.1 v-Bases versus k-Counterexamples

We now analyze the tight relationships between v-bases and k-counterexamples. First we introduce the notion of partial v-basis, which formalizes the uncomplete knowledge of a v-basis.

Definition 6.3 A *partial vaccines basis* is a collection $A_1, \ldots, A_k$ (k an ordinal) of vaccines giving pairwise different maximal criteria. □

Lemma 6.4 *Every partial vaccines basis $\{A_1, \ldots, A_k\}$ is a k-counterexample.*

As a corollary, we get that every v-basis $\{A_1, \ldots, A_k\}$ is a k-counterexample. The next important result shows that also the other direction holds, thus providing a way to find the v-bases:

Theorem 6.5 *If the modular dimension is $k < \omega$, then every k-counterexample is a v-basis.*

Combining these results, we get the following characterization of the v-bases:

Corollary 6.6 (Characterization) *If the modular dimension is $k < \omega$, then the v-bases are exactly the k-counterexamples.*

Therefore, the above results suggest a way to find the optimal criteria: seek for vaccines produced by objects in k-counterexamples.

In fact, Theorem 6.5 says much more: if we know that the m-dimension is $k < \omega$ (e.g. via Corollary 4.5), then a v-basis is automatically provided by a k-counterexample.

Another immediate consequence of Theorem 6.5 is about the existence of v-bases:

Corollary 6.7 *If the modular dimension is $k < \omega$, there is a v-basis.*

In order to effectively find a v-basis, Theorem 6.5 requires the knowledge of the m-dimension, which as said can be computed using Corollary 4.5. Anyway, there is another fundamental result that, starting from a not complete knowledge of it (a partial v-basis), ensures that we have found a v-basis:

Theorem 6.8 (Covering) *Let $A_1, \ldots, A_k$ ($k < \omega$) be a partial v-basis. It is a v-basis iff every healthy object belongs to at least one $\mathbf{V}_{A_i}$: $\cup_{i \in [1,k]} \mathbf{V}_{A_i} = \mathcal{P}$ (i.e. the criteria 'cover' the healthy objects).*

The above theorem thus provides an alternative powerful methodology to find a v-basis: build up a k-counterexample with k as great as possible; prove that its elements are vaccines (Theorem 6.5); check if the criteria cover the healthy objects (Theorem 6.8).

We will later (Section 9) successfully employ this methodology in the applications of the theory to term rewriting.

7 Disjunctive Criteria

The notion of criterion for modularity that we have given in Definition 3.1 is not the only one which has been studied. Another kind of criteria, e.g. studied in [19, 27], requires only one of the objects to be constrained in order to ensure their combination is healthy. So, we introduce this concept:

Definition 7.1 $\mathcal{Q}$ is a *disjunctive criterion (for the $\odot$-modularity of $\mathcal{P}$)*, or *d-criterion* for short, if $\forall T_1, \ldots, T_n . T_1 \in \mathcal{P} \vee \ldots \vee T_n \in \mathcal{P} \Rightarrow T_1 \odot \ldots \odot T_n \in \mathcal{P}$. □

The motivation for the adjective 'disjunctive' should be clear from the definition; analogously, the usual criterion of Definition 3.1 could be dubbed 'conjunctive'.

Unlike the standard criteria, the d-criteria space is linearly ordered, since only one object instead of all objects is constrained. The following definition formalizes the top object in this space:

Definition 7.2 The *kernel* $\mathcal{K}$ is the greatest disjunctive criterion, that is $\mathcal{K} = \{T \in \mathcal{P} : \forall T' \in \mathcal{P}.\, T \odot T' \in \mathcal{P} \ni T' \odot T\}$. □

Nicely, from a v-basis we can obtain right away the kernel:

Theorem 7.3 *Suppose* $\{A_i\}_{i=1\ldots k}$ *is a vaccines basis. Then the kernel is* $\cap_{i=1\ldots k} \mathbf{V}_{A_i}$.

Rather interestingly, the kernel has an important algebraic meaning, as the following lemma reveals:

Lemma 7.4 $\mathcal{K} = 0/_{=_\mathcal{P}}$.

That is to say, the kernel is just the class of $=_\mathcal{P}$ *-neutral elements*.

8 Counterexample Structures

In this section we turn our attention to the other side of the coin: when modularity fails. We formally study what happens when two objects give a counterexample to modularity.

Definition 8.1 A couple of classes $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ is a *counterexample structure* (*c-structure*), (w.r.t. $\odot$ and $\mathcal{P}$) if in every counterexample one of the two objects belongs to $\mathcal{Q}_1$ and the other to $\mathcal{Q}_2$. □

The canonical ordering on structures is: $\{\mathcal{Q}_1, \mathcal{Q}_2\} \subseteq_{struct} \{\mathcal{Q}'_1, \mathcal{Q}'_2\}$ iff $(\mathcal{Q}_1 \subseteq \mathcal{Q}'_1 \wedge \mathcal{Q}_2 \subseteq \mathcal{Q}'_2) \vee (\mathcal{Q}_1 \subseteq \mathcal{Q}'_2 \wedge \mathcal{Q}_2 \subseteq \mathcal{Q}'_1)$. Then, we say that a structure $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ is better than another structure $\{\mathcal{Q}'_1, \mathcal{Q}'_2\}$ if $\{\mathcal{Q}_1, \mathcal{Q}_2\} \subset_{struct} \{\mathcal{Q}'_1, \mathcal{Q}'_2\}$: this means we can provide with $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ a more precise (smaller) description than with $\{\mathcal{Q}'_1, \mathcal{Q}'_2\}$. The best structure is so the minimum w.r.t. $\subseteq_{struct}$.

From a v-basis we can recover the *best* counterexample structure, as the next result shows:

Theorem 8.2 *If* $\{A_1, A_2\}$ *is a vaccines basis, then* $\{\neg \mathbf{V}_{A_1} \wedge \mathcal{P}, \neg \mathbf{V}_{A_2} \wedge \mathcal{P}\}$ *is the best counterexample structure.*

Analogous results can be stated for v-bases of higher dimension.

9 Applications to Term Rewriting

We now provide some applications of the theory to the study of modularity for Term Rewriting Systems.

So, we let $\mathcal{O}$ =TRSs and consider as usual the combination operator $\odot$ to be the disjoint sum ($\oplus$) of two TRSs: when the signatures overlap the TRSs are renamed to get disjoint signatures, and then their (disjoint) union is taken.

It is trivial to see that (TRSs, $\oplus$) is an aci-groupoid. Hence to show that (TRSs, $\oplus$) is acid w.r.t. $\mathcal{P}$ (cf. Subsection 3.1) it suffices to prove the $=_\mathcal{P}$ -idempotency.

All the main properties of TRSs are modular, but for three: termination, completeness and uniqueness of normal forms. In the following of this section we investigate all these properties.

9.1 Termination

We let $\mathcal{O}$ =TRSs, $\odot = \oplus$, and the healthiness property $\mathcal{P}$ =Termination (Termination will be also indicated with the acronym SN, after Strong Normalization).

Lemma 9.1 (*TRSs*, $\oplus$) *is SN-acid.*

Among the many results on the modularity of termination (see e.g. [20, 13, 22, 24] for a panoramic), the best results so far obtained are the ones in [21] and [14]. We will come back to the result of [14] in the next subsection. In [21] Ohlebusch, generalizing a previous result of Gramlich for finitely branching TRSs ([10]), proved that '$\mathcal{C}_\mathcal{E}$-termination' is modular. It is straightforward to see that the class of $\mathcal{C}_\mathcal{E}$-terminating TRSs coincides with the class of TRSs vaccinated via $\{or(X,Y) \to X, or(X,Y) \to Y\}$. This, a posteriori, implies that the above TRS is a vaccine (for the modularity of termination).

Hence, using Theorem 6.1 we obtain right away:

Theorem 9.2 *$\mathcal{C}_\mathcal{E}$-termination is a maximal criterion.*

That is to say, the result of [21] *cannot* be improved.

But what is the complexity of the modular space for termination? The following result gives a formal confirmation that the topic is quite intricated:

Theorem 9.3 *The m-dimension is at least three.*

The proof of the above result makes use of Lemma 4.3.

Whether the m-dimension is indeed three, is still one of the most important open problems (we conjecture it is).

9.1.1 The Left-Linear Case

As just seen, the situation for termination is quite complicated, since we have proved that the m-dimension is at least three, and only one vaccine has been found so far. In the left-linear case we will be able to *completely solve* the problem, finding a v-basis.

There are two best results on the modularity of termination for left-linear TRSs. The first stems from the one seen above: in the left-linear case, $\{or(X,Y) \to X, or(X,Y) \to Y\}$ is a vaccine.

So, by Theorem 6.1 we can infer that $\mathcal{C}_\mathcal{E}$-termination is a maximal criterion even for left-linear TRSs.

The second is the result proved in [14]. Recall that a TRS is said consistent (with respect to reduction), briefly $\mathrm{CON}^\to$, if no term reduces to two different variables. In the aforementioned paper it has been shown that termination is modular for left-linear and consistent TRSs.

We have seen in Section 4 that there are deep relationships between k-counterexamples and v-bases. The most famous counterexample to the modularity of termination has been given by Toyama in [25]: $\{F(0,1,X) \to F(X,X,X)\}$ and $\{or(X,Y) \to X, or(X,Y) \to Y\}$. As seen above, $\{or(X,Y) \to X, or(X,Y) \to Y\}$ is a vaccine. Hence, a stimulating hypothesis is that $\{F(0,1,X) \to F(X,X,X)\}$ is a vaccine as well. Amazingly, this turns out to be true:

Theorem 9.4 *For left-linear TRSs,* $\mathbf{V}_{\{F(0,1,X)\to F(X,X,X)\}} = \mathrm{SN} \wedge \mathrm{CON}^\to$.

That is to say, the class of left-linear TRSs vaccinated by $\{F(0,1,X) \to F(X,X,X)\}$ is just the criterion found in [14].

Corollary 9.5 *In the left-linear case, $\{F(0,1,X) \to F(X,X,X)\}$ is a vaccine.*

Hence, we get

Corollary 9.6 *In the left-linear case, $\mathrm{SN} \wedge \mathrm{CON}^\to$ is a maximal criterion.*

Thus, the result of [14] *cannot* be improved.

The remarkable thing is that with these two vaccines we have completed the analysis of the modular space, since they form a v-basis:

Theorem 9.7 *The m-dimension for left-linear TRSs is two, and a vaccines basis is given by* $\{F(0,1,X) \to F(X,X,X)\}$, $\{or(X,Y) \to X, or(X,Y) \to Y\}$.

That is to say, the above two optimal criteria completely solve the problem of modularity of termination for left-linear TRSs: there are *no other* optimal criteria and *all the other* criteria are subsumed by one of the two.

Also, being the m-dimension 2, by Corollary 6.6 we have a characterization of the v-bases: they are just the counterexamples.

As far as d-criteria are concerned, Middeldorp in [19] showed that whenever one of two terminating TRSs is both non-collapsing and non-duplicating, then their disjoint sum is terminating; that is to say, he proved that "terminating and non-collapsing and non-duplicating" is a disjunctive criterion. Toyama, Klop and Barendregt showed in [27] that whenever one of two terminating TRSs is confluent and non-collapsing, then their disjoint sum is terminating (hence, they proved that "terminating and confluent and non-collapsing" is a d-criterion).

Using the result on d-criteria (Theorem 7.3), we can properly generalize both of these results in the left-linear case, giving the *best* d-criterion (the kernel):

Theorem 9.8 *For left-linear TRSs,* $\mathrm{CON}^{\to} \wedge \mathcal{C}_{\mathcal{E}}$*-termination is the greatest disjunctive criterion for the modularity of termination.*

We now consider c-structures. Ohlebusch in [21] (again, extending a result of Gramlich in [10] for finitely branching TRSs), showed that in every counterexample one of the TRSs is not $\mathcal{C}_{\mathcal{E}}$-terminating and the other is collapsing (hence, in our terminology, he showed that { $\mathcal{C}_{\mathcal{E}}$-termination, non-collapsibility } is a c-structure). Schmidt-Schauß, Marchiori and Panitz showed in [24] that, in the left-linear case, in every counterexample one of the TRSs is $\mathrm{CON}^{\to}$ and the other is $\neg\mathrm{CON}^{\to}$ (that is, { $\mathrm{CON}^{\to}$, $\neg\mathrm{CON}^{\to}$ } is a c-structure). Both of these results require a not easy proof. Via Theorem 8.2, we can easily not only generalize all of these results in the left-linear case, but also provide the *best* c-structure:

Theorem 9.9 $\{\neg\mathrm{CON}^{\to} \wedge \mathrm{SN}, \neg\mathcal{C}_{\mathcal{E}}$*-termination* $\wedge \mathrm{SN}\}$ *is the best counterexample structure.*

The above theorem gives the following result: *in every counterexample to the modularity of termination, one of the TRSs is non consistent and the other is non* $\mathcal{C}_{\mathcal{E}}$*-terminating.*

Other applications, as mentioned in Section 6, include the possibility to give easy proofs of all the previously existing results on modularity. In the full paper it is for example shown how all the results in [23] and [19] can be given a straightforward proof.

Finally, the optimality of the v-basis allows to infer right away results on the relative strength of other criteria.

For instance, it has been directly proved with some effort in [10] that Simple Termination implies $\mathcal{C}_{\mathcal{E}}$-termination, and that termination plus non-duplication imply $\mathcal{C}_{\mathcal{E}}$-termination. These results immediately follow from Theorem 9.2, once noticed that Simple Termination ([13]) and termination plus non-duplication ([23]) are criteria, and that $\{or(X,Y) \to X, or(X,Y) \to Y\}$ is both simply terminating and non-duplicating.

10 The Other Cases

Two other major properties are missing. Using vaccines, we will completely solve the problem of their modularity, providing for both of them a v-basis. For lack of space, here we just hint at the main results: the presentation of the v-basis. As seen before, one can then derive corresponding results on d-criteria, c-structures and so on.

10.1 Completeness

Let $\mathcal{O}$ =TRSs, $\odot = \oplus$, and $\mathcal{P}$ =Completeness (recall that completeness is termination plus confluence).

So far, there are four best results on the modularity of completeness. Via a result of Toyama ([25]), Rusinowitch proved in [23] that completeness is modular for non-collapsing TRSs and for non-duplicating TRSs. Gramlich in [8] proved that completeness is modular for locally confluent overlay TRSs. Toyama, Klop and Barendregt showed in their famous paper [26] (see also the full version [27]) that completeness is modular for left-linear TRSs.

The following main result not only improves on all these papers, but *completely solves* the problem of the modularity of completeness:

Theorem 10.1

$$\left\{ \begin{array}{l} F(0,1,X) \to F(X,X,X) \\ F(X,Y,Z) \to 2 \\ 0 \to 2 \\ 1 \to 2 \end{array} \right\} \qquad \left\{ \begin{array}{l} h(X,Y,Y) \to X \\ h(X,X,Y) \to Y \end{array} \right\}$$

is a vaccines basis for the modularity of completeness.

10.2 Uniqueness of Normal Forms

Let $\mathcal{O}$ =TRSs, $\odot = \oplus$, and $\mathcal{P}$ =UN$^{\to}$ (recall that a TRS is said to have unique normal forms, briefly UN$^{\to}$, if every term has at most one normal form).

Two best results are so far known on the modularity of UN$^{\to}$. In [18] it has been proved that the equational uniqueness of normal forms (UN) is a criterion for the modularity of UN$^{\to}$. In [16] it has been proved the modularity of UN$^{\to}$ for left-linear TRSs (a long-standing open problem, cf. [6]).

The following theorem not only improves on these results, but *completely solves* the problem of the modularity of UN$^{\to}$:

Theorem 10.2

$$\left\{ \begin{array}{l} a \to c \\ a \to e \\ e \to e \\ b \to e \\ b \to d \end{array} \right\} \quad \left\{ \; F(X,X) \to A \; \right\}$$

is a vaccines basis for the modularity of UN$^{\to}$.

11 Remarks

In this extended abstract we have sketched the core of the theory of vaccines, and presented as a particular instance some successful applications to modularity in term rewriting. For reasons of space and clarity, other interesting results have been omitted. In particular, besides many other technical results which are variations of the main results here presented, a major topic was neglected (cf. [15] for a complete exposition), namely *multimodularity*, where other combinations of more than two objects are studied. Again, via a v-basis we can obtain precise information on what kind of multimodular behaviour a certain property satisfies.

Currently, we are investigating practical applications of the theory to the study of modularity for other paradigms, like functional or logic programming (cf. [4]). Note that even in the rewriting field there are still many other modularity topics to which the theory of vaccines can be applied, including e.g. more involved combinations of TRSs (like composable ones, cf. [22] for a survey),

higher order rewriting in its various forms (see e.g. [12, 11]), conditional rewriting ([12, 20]), combinations with λ-calculus and systems in the λ-cube (cf. [3, 1, 2, 7]), and so on. For instance, we have applied the theory to the criterion developed in [9] for conditional rewriting, showing that it is optimal. Also, using unraveling theory (cf. [17]) we have been able to automatically translate a lot of modularity results from term rewriting to conditional rewriting: for instance, we have lifted the result of Theorem 9.7, showing that, for left-linear normal CTRSs, the same two TRSs provide a v-basis for decreasingness.

Acknowledgments

I wish to thank Jan Willem Klop and Aart Middeldorp for scrutinizing a previous version of this paper.

References

[1] F. Barbanera and M. Fernandez. Modularity of termination and confluence in combinations of rewrite systems with λ_ω. In *20th ICALP*, volume 700 of *LNCS*. Springer–Verlag, 1993.

[2] F. Barbanera, M. Fernandez, and H. Geuvers. Modularity of strong normalization and confluence in the algebraic λ-cube. In *Proceedings Nineth IEEE Symposium on Logic in Computer Science*, pages 406–415, 1994.

[3] V. Breazu-Tannen and J. Gallier. Polymorphic rewriting conserves algebraic strong normalization. *Theoretical Computer Science*, 83:3–28, 1990.

[4] A. Brogi, P. Mancarella, D. Pedreschi, and F. Turini. Modular logic programming. *ACM TOPLAS*, 16(4), 1994.

[5] N. Dershowitz and J.-P. Jouannaud. Rewrite systems. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, chapter 6, pages 243–320. Elsevier – MIT Press, 1990.

[6] N. Dershowitz, J.-P. Jouannaud, and J.W. Klop. Open problems in rewriting. In R.V. Book, editor, *Proceedings of the Fourth International Conference on Rewriting Techniques and Applications*, volume 488 of *LNCS*, pages 445–456. Springer–Verlag, 1991.

[7] R. Di Cosmo and D. Kesner. Combining first order algebraic rewriting systems, recursion and extensional lambda calculi. In *21st ICALP*, LNCS. Springer–Verlag, 1994.

[8] B. Gramlich. Relating innermost, weak, uniform and modular termination of term rewriting systems. In *International Conference on Logic Programming and Automated Reasoning*, volume 624 of *LNAI*, pages 285–296. Springer-Verlag, 1992.

[9] B. Gramlich. Sufficient conditions for modular termination of conditional term rewriting systems. In *Third International Workshop on Conditional Term Rewriting Systems*, volume 656 of *LNCS*, pages 128–142. Springer-Verlag, 1993.

[10] B. Gramlich. Generalized sufficient conditions for modular termination of rewriting. *Applicable Algebra in Engineering, Communication and Computing*, 5:131–158, 1994.

[11] J.-P. Jouannaud and M. Okada. Executable higher-order algebraic specification languages. In *Proceedings of the Sixth IEEE Symposium on Logic in Computer Science*, pages 350–361, 1991.

[12] J.W. Klop. Term rewriting systems. In S. Abramsky, Dov M. Gabbay, and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 2, chapter 1, pages 1–116. Clarendon Press, Oxford, 1992.

[13] M. Kurihara and A. Ohuchi. Modularity of simple termination of term rewriting systems. *Journal of IPS Japan*, 31(5):633–642, 1990.

[14] M. Marchiori. Modularity of completeness revisited. In J. Hsiang, editor, *Proceedings of the Sixth International Conference on Rewriting Techniques and Applications*, volume 914 of *LNCS*, pages 2–10. Springer–Verlag, 1995.

[15] M. Marchiori. The theory of vaccines. Technical Report 27, Dept. of Pure and Applied Mathematics, University of Padova, 1995.

[16] M. Marchiori. On the modularity of normal forms in rewriting. *Journal of Symbolic Computation*, 22:143–154, 1996.

[17] M. Marchiori. Unravelings and ultra-properties. In *Proceedings of the Fifth International Conference on Algebraic and Logic Programming (ALP'96)*, volume 1139 of *LNCS*, pages 107–121. Springer–Verlag, 1996.

[18] A. Middeldorp. Modular aspects of properties of term rewriting systems related to normal forms. In N. Dershowitz, editor, *Proceedings of the Third International Conference on Rewriting Techniques and Applications*, volume 355 of *LNCS*, pages 263–277. Springer–Verlag, April 1989.

[19] A. Middeldorp. A sufficient condition for the termination of the direct sum of term rewriting systems. In *Proceedings of the Fourth IEEE Symposium on Logic in Computer Science*, pages 396–401, 1989.

[20] A. Middeldorp. *Modular Properties of Term Rewriting Systems.* PhD thesis, Vrije Universiteit, Amsterdam, November 1990.

[21] E. Ohlebusch. On the modularity of termination of term rewriting systems. *Theoretical Computer Science*, 136(2):333–360, 1994.

[22] E. Ohlebusch. Modular properties of composable term rewriting systems. *Journal of Symbolic Computation*, 20(1):1–41, 1995.

[23] M. Rusinowitch. On termination of the direct sum of term rewriting systems. *Information Processing Letters*, 26:65–70, 1987.

[24] M. Schmidt-Schauß, M. Marchiori, and S.E. Panitz. Modular termination of r-consistent and left-linear term rewriting systems. *Theoretical Computer Science*, 149(2):361–374, 1995.

[25] Y. Toyama. On the Church-Rosser property for the direct sum of term rewriting systems. *Journal of the ACM*, 1(34):128–143, 1987.

[26] Y. Toyama, J.W. Klop, and H.P. Barendregt. Termination for the direct sum of left-linear term rewriting systems. In N. Dershowitz, editor, *Proceedings of the Third International Conference on Rewriting Techniques and Applications*, volume 355 of *LNCS*, pages 477–491, Chapel Hill, April 1989. Springer–Verlag. Extended version: Report CS-R8923, CWI, Amsterdam.

[27] Y. Toyama, J.W. Klop, and H.P. Barendregt. Termination for direct sums of left-linear complete term rewriting systems. *Journal of the ACM*, 42(6):1275–1304, November 1995.